Gromov-Witten theory relative to an SNC divisor

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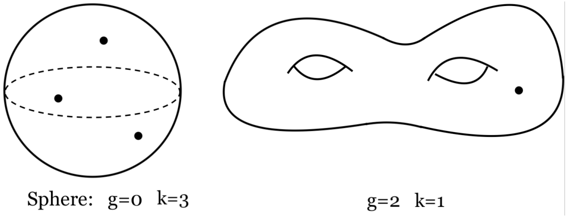
Geometry/Topology Seminar April 2019

Holomorphic curves

- X: smooth complex proj variety OR symplectic manifold with an almost complex structure J ($J\colon TX \longrightarrow TX$, $J^2 = -\text{id}$)
- Genus g curve Σ with k distinct marked points $\vec{z} = (z_1, \ldots, z_k)$



- Degree $A \in H_2(X, \mathbb{Z})$

$$\Rightarrow \mathcal{M}_{g,k}(X, A) = \Big\{ (u, \Sigma, \vec{z}) : \Sigma \xrightarrow{u} X,\ \bar{\partial} u = 0,\ [u(\Sigma)] = A \Big\} / \sim$$

Relative case

- SNC divisor in X: $D = \bigcup_{i=1}^{N} D_i$

- **Remark.** In Symp case, general definition of SNC divisor is due to McLean, Zinger, and I (2014). We need J that is compatible with both ω and D, plus more conditions

- Tangency orders: $\mathfrak{s} = (s_1, \ldots, s_k)$

$$s_a = (s_{ai})_{i=1}^{N} \in \mathbb{N}^N, \qquad A \cdot D_i = \sum_{a=1}^{k} s_{ai}$$

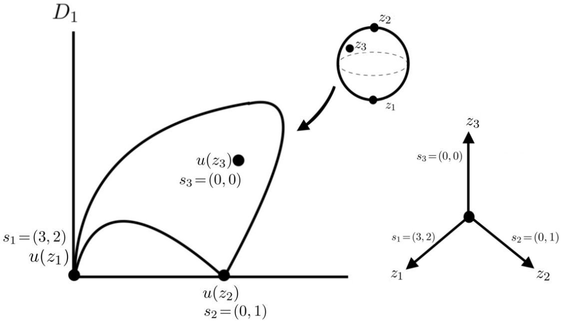
- $\mathcal{M}_{g,k}(X, A) \supset \mathcal{M}_{g,\mathfrak{s}}(X, D, A) =$

$$\Big\{ [u, \Sigma, \vec{z}] \colon u^{-1}(D) \subset \{z_1, \ldots, z_k\},\ \mathsf{ord}_{z_a}(u, D_i) = s_{ai} \Big\}$$

- $\text{Exp-dim}_{\mathbb{C}} \mathcal{M}_{g,\mathfrak{s}}(X, D, A) = c_1^{TX}(A) + (n-3)(1-g) + k - A \cdot D$

Example

- $X = \mathbb{P}^2$, $D = D_1 \cup D_2$ union of two hyperplanes (lines)
- $A = [3] \in H_2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$, $g = 0$, $k = 3$
- $\mathfrak{s} = \big(s_1 = (3,2), s_2 = (0,1), s_3 = (0,0)\big)$
- $u\big([z,w]\big) = [z^3, z^2w, p(z,w)]$, $\dim_{\mathbb{C}} \mathcal{M}_{0,\mathfrak{s}}(\mathbb{P}^2, D, [3]) = 5$



GOAL (analytical approach)

- Compactify $\mathcal{M}_{g,\mathfrak{s}}(X,D,A) \rightsquigarrow \overline{\mathcal{M}}_{g,\mathfrak{s}}(X,D,A)$
- Set up the deformation theory
- Construct VFC

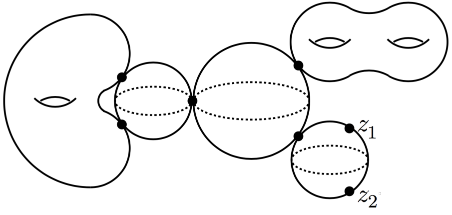
$\Rightarrow$ we can define Gromov-Witten invariants of (X,D):
count k-marked genus g degree A holomorphic curves of tangency type $\mathfrak{s}$ with D plus other conditions on the domain or the image

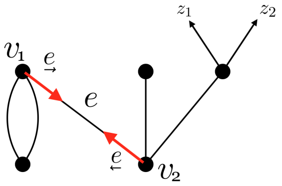
$$\begin{array}{ccc} \mathcal{M}_{g,\mathfrak{s}}(X,D,A) & \hookrightarrow & \mathcal{M}_{g,k}(X,A) \\ \big\downarrow & & \big\downarrow \text{stable compactification} \\ ? & \hookrightarrow & \overline{\mathcal{M}}_{g,k}(X,A) \end{array}$$

Compactification (classical case)

- **Stable** (Gromov) compactification $\overline{\mathcal{M}}_{g,k}(X,A)$: holomorphic maps with nodal domain and finite automorphism

A nodal curve of genus 4 with 2 marked points





Decorated dual graph Γ

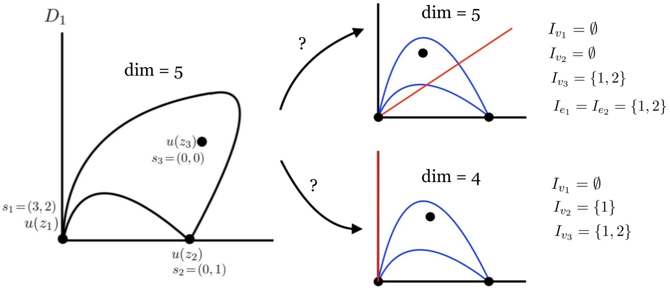
- Vertices $v \in \mathbb{V} \rightsquigarrow$ irreducible components Σ_v
- Edge $e \in \mathbb{E} \rightsquigarrow$ nodes q_e connecting $q_{\underset{\rightarrow}{e}} \in \Sigma_{v_1}$ and $q_{\underset{\leftarrow}{e}} \in \Sigma_{v_2}$
- Flags (open edges) $\rightsquigarrow$ marked points
- Decorations: $\mathbb{V} \ni v \rightsquigarrow$ <u>genus g_v</u> of Σ_v, <u>degree A_v</u> of $u_v = u|_{\Sigma_v}$
- Extra decorations in the relative case:

$$a\text{-th marked point } z_a \rightsquigarrow s_a \in \mathbb{N}^N$$

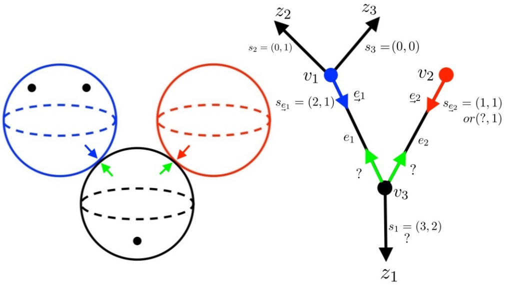
$$\mathbb{E} \ni e \rightsquigarrow I_e \subset \{1, \ldots, N\}, \quad u(q_e) \subset D_{I_e} = \bigcap_{i \in I_e} D_i$$

$$\mathbb{V} \ni v \rightsquigarrow I_v \subset \{1, \ldots, N\}, \quad \mathrm{Im}(u_v = u|_{\Sigma_v}) \subset D_{I_v}$$

Example



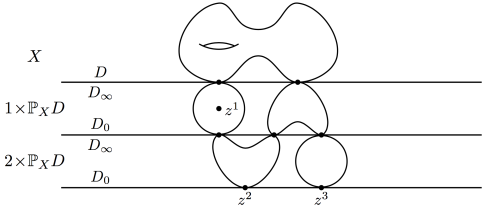
D_1
dim = 5
$u(z_3)$
$s_3 = (0,0)$
$s_1 = (3,2)$
$u(z_1)$
$u(z_2)$
$s_2 = (0,1)$
?
?
dim = 5
$I_{v_1} = \emptyset$
$I_{v_2} = \emptyset$
$I_{v_3} = \{1,2\}$
$I_{e_1} = I_{e_2} = \{1,2\}$
dim = 4
$I_{v_1} = \emptyset$
$I_{v_2} = \{1\}$
$I_{v_3} = \{1,2\}$



z_2
z_3
$s_2 = (0,1)$
$s_3 = (0,0)$
v_1
v_2
$s_{\underset{\leftarrow}{e}_1} = (2,1)$
$\underset{\rightarrow}{e}_1$
$\underset{\rightarrow}{e}_2$
$s_{\underset{\leftarrow}{e}_2} = (1,1)$
$or(?,1)$
e_1
e_2
?
?
v_3
$s_1 = (3,2)$
?
z_1

Previous works (smooth D, early 2000)

- Jun Li (algebraic), Ionel-Parker and Li-Ruan (symplectic)
  - **idea:** In order to construct a (so called **relative**) compactification, they also degenerate the target



  - **issue 1:** Changing the target makes the analysis hard (so details still incomplete after 15 years)
  - **issue 2:** It does not generalize to SNC case

Previous works (SNC D and more, mid 2000-current)

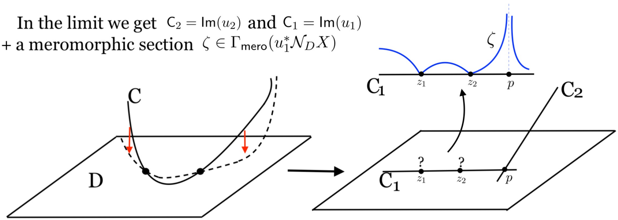
- Gross-Siebert, Abramovich-Chen, ... (algebraic case)
  - **idea:** They consider pairs of holomorphic maps and maps between certain **sheaves of monoids** on domains and a fixed sheaf of monoids on the target
  - **issue 1:** complicated for computations
  - **issue 2:** specific to the algebraic category
- Brett Parker (analytical, certain **almost Kähler** cases)
  - **idea:** Similarly, pairs of holomorphic maps and maps of certain analytical sheaves
  - **issue 1:** very complicated (involves non-Hausdorff spaces)
  - **issue 2:** it essentially works in the Kähler category

Observation 1

- in the limit, for each $u_v\colon \Sigma_v \longrightarrow D_{I_v}$ and every $i \in I_v$, we get a meromorphic section

$$\zeta_{v,i} \in \Omega_{\mathsf{mero}}(u^* \mathcal{N}_X D_i)$$

with zeros/poles only at marked or nodal points



- $\zeta_{v,i}$ is well-defined up to action of $\mathbb{C}^*$

Observation 2

- A tuple $\big(u_v\colon \Sigma_v \longrightarrow D_{I_v}, \zeta_v = (\zeta_{v,i})_{i\in I_v}\big)$ allows us to define tangency order of each point as a vector in $\mathbb{Z}^N$

$$\mathrm{ord}_{u_v,\zeta_v}(x) = \big(\mathrm{ord}^i_{u_v,\zeta_v}(x)\big)_{i=1}^N \in \mathbb{Z}^N \qquad \forall\ x \in \Sigma_v$$

$$\mathrm{ord}^i_{u_v,\zeta_v}(x) = \begin{cases} \mathrm{ord}_x(u_v, D_i) & \text{if } i \notin I_v \\ \mathrm{ord}_x(\zeta_{v,i}) & \text{if } i \in I_v \end{cases}$$

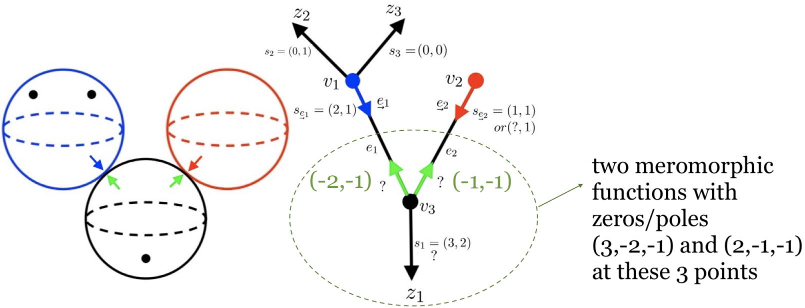
- **Definition:** A pre-log map is a tuple

$$f = \big(u_v\colon (\Sigma_v, \vec{z}_v) \longrightarrow D_{I_v}, \zeta_v = (\zeta_{v,i})_{i\in I_v}\big)_{v\in\mathbb{V}}$$

such that

1. order at the a-th marked point is s_a
2. at each node $q_e = (q_{\underset{\rightarrow}{e}} \sim q_{\underset{\leftarrow}{e}})$, orders are dual: $s_{\underset{\rightarrow}{e}} = -s_{\underset{\leftarrow}{e}}$
3. at any other point, order is trivial

Back to the Example



In case 2, on Σ_{v_2} we need one holomorphic section of $\mathcal{N}_{\mathbb{P}^2} D_1 = \mathcal{O}(1)$ with a simple zero at $q_{\underrightarrow{e}_2}$: there is exactly one such section up to action of $\mathbb{C}^*$

$\star$: We need more conditions as case 1 is still OK

A linear map associated to the decorated dual graph

- Extra decoration: $\underset{\rightarrow}{e} \rightsquigarrow s_{\underset{\rightarrow}{e}} \in \mathbb{Z}^N$ at each nodal point
- After fixing random orientations on edges:

$$\rho : \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \longrightarrow \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e}$$

$$1_e \longrightarrow \begin{cases} s_{\underset{\rightarrow}{e}} & e\text{−th component} \\ 0 & \text{otherwise} \end{cases}$$

$$1_{v,i} \longrightarrow \begin{cases} 1_{e,i} & \text{if } \underset{\rightarrow}{e} \text{ begins with } v \\ -1_{e,i} & \text{if } \underset{\rightarrow}{e} \text{ ends at } v \\ 0 & \text{otherwise} \end{cases}$$

- **Claim:** some part of the Def-Obs information is encoded

$$\mathsf{K} = \ker(\rho_{\mathbb{R}}) \qquad \mathsf{Ck} = \operatorname{coker}(\rho_{\mathbb{C}})$$

These are <u>independent</u> of the orientation we chose to define ρ

Def-Obs information encoded in K and Ck

- Inside K we consider the cone

$$\sigma = \mathsf{K} \cap (\mathbb{R}_{\geq 0}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{R}_{\geq 0}^{I_v})$$

  which possibly could be just the origin!

- Corresponding to Ck we get the complex Lie group

$$\mathcal{G} = \exp(\mathsf{Ck}) := \frac{\prod_{e \in \mathbb{E}} (\mathbb{C}^*)^{I_e}}{(\mathbb{C}^*)^{\mathbb{E}} \times \prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v}}$$

- **Lemma:** Corresponding to every pre-log map f with the decorated dual graph Γ we get an element $\mathrm{ob}(f) \in \mathcal{G}$

Log holomorphic curves

- **Definition**. Define $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ to be the space of equivalence classes of stable pre-log maps

$$f = \big(u_v\colon (\Sigma_v, \vec{z}_v) \longrightarrow D_{I_v}, \zeta_v = (\zeta_{v,i})_{i\in I_v}\big)_{v\in\mathbb{V}}$$

  where

  1. $\exists\ s_v \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v} \subset \mathbb{R}^N$ and positive numbers $\lambda_e \in \mathbb{R}_+$ such that

$$s_{v_2} - s_{v_1} = \lambda_e s_{\underset{\rightarrow}{e}}$$

     for all oriented edges $\underset{\rightarrow}{e}$ from v_1 to v_2

  2. $\mathrm{ob}(f) = 1$

- Define

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A) = \bigcup_{\Gamma} \mathcal{M}_{g,\mathfrak{s}}(X,D,A)_\Gamma$$

Remarks

- Condition 1 is equivalent to σ being a maximal convex rational polyhedral cone in K
- In the works of ACGS, Condition 1 is equivalent to the existence of certain tropical curves
- Comparing to relative GW theory, the vectors s_v in Condition 1 give N-partial orderings on the irreducible components relative to $D_1, \ldots, D_N$
- From σ we get an affine toric variety Y_σ with the fan σ
- **Lemma:** The space of gluing parameters (up to some multiplicity) is a neighborhood of 0 in Y_σ

$$\text{codim}_{\mathbb{C}}\, \mathcal{M}_{g,\mathfrak{s}}(X, D, A)_\Gamma = \dim \mathsf{K}$$

- Condition 2 has no explicit analogue in the literature

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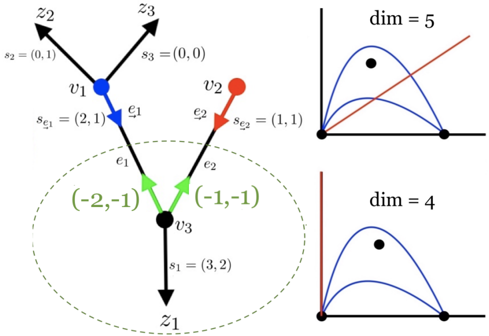

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$I_{v_2} = \emptyset$

$I_{v_3} = \{1,2\}$

$I_{e_1} = I_{e_2} = \{1,2\}$

$s_{v_1} = s_{v_2} = (0,0)$

$\lambda_{e_1} s_{\underrightarrow{e}_1} = \lambda_{e_2} s_{\underrightarrow{e}_2} = s_{v_3}$

Not possible!

$I_{v_1} = \emptyset$

$I_{v_2} = \{1\}$

$I_{v_3} = \{1,2\}$

$s_{v_1} = (0,0)$ $s_{v_2} = (1,0)$

$s_{v_3} = (2,1)$

$s_{\underrightarrow{e}_1} = s_{v_3} - s_{v_1}$ $s_{\underrightarrow{e}_2} = s_{v_3} - s_{v_2}$

– So the first configuration does not satisfy condition 1

– Second one does and $\mathcal{G}$ is trivial (nothing to check in Condition 2)

- **Compactness Theorem (–, 2017)** For suitable choice of J (including holomorphic case)
  - $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is compact, metrizable, and of the expected $\mathbb{C}$-dimension

    $$c_1^{TX}(A)+(n-3)(1-g)+k-A\cdot D = c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k$$

  - the natural forgetful map

    $$\overline{\mathcal{M}}_{g,\mathfrak{s}}(X,D,A) \longrightarrow \overline{\mathcal{M}}_{g,k}(X,A)$$

    is an embedding if $g=0$, and it is locally an embedding (like an immersion) if $g>0$
- **Proposition (–, 2017)** If D is smooth, there is a surjective map

  $$\overline{\mathcal{M}}^{\mathsf{rel}}_{g,\mathfrak{s}}(X,D,A) \longrightarrow \overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$$

Transversality (deformation theory): classical case

- For a fixed Σ, ∞-dimensional bundle

$$\mathcal{E} \longrightarrow \mathcal{B} = \mathsf{Map}(\Sigma, X), \qquad \mathcal{E}_u = \Omega^{0,1}(u^*TX)$$

- $\bar\partial\colon \mathcal{B} \longrightarrow \mathcal{E}$ is a smooth section
- zero set of $\bar\partial$ is the set of holomorphic maps from Σ into X
- $D_u\bar\partial\colon \Omega^0(u^*TX) \longrightarrow \Omega^{0,1}(u^*TX)$: linearization of $\bar\partial$ at u
- $D_u\bar\partial = \bar\partial_{\mathsf{std}} +$ compact perturbation, and

$$\mathsf{Def}(u) = \ker(D_u\bar\partial)$$
$$\mathsf{Obs}(u) = \mathsf{coker}(D_u\bar\partial)$$

are finite dimensional

- By Riemann-Roch

$$\dim_{\mathbb{R}}\mathrm{Def}(u) - \dim_{\mathbb{R}}\mathrm{Obs}(u) = 2(c_1^{TX}(A) + n(1-g))$$

- If $D_u\bar{\partial}$ is surjective ($\mathrm{Obs}(u) = 0$) $\Rightarrow$ <u>Around u</u>, $\mathcal{M}_{g,k}(X,A)$ is a smooth orbifold of real dimension

$$2\big(c_1^{TX}(A) + (n-3)(1-g) + k\big)$$

- The idea around the transversality issue is to consider global (Ruan-Tian) or local (Li-Tian, Fukaya-Ono, etc) deformations $\bar{\partial}u = \nu$ of $\bar{\partial}$-section

- For $f = (u, \Sigma, \vec{z})$

$$\begin{aligned} 0 \longrightarrow \mathsf{Def}(u) \longrightarrow \mathsf{Def}(f) \longrightarrow \mathsf{Def}(\Sigma, \vec{z}) \\ \longrightarrow \mathsf{Obs}(u) \longrightarrow \mathsf{Obs}(f) \longrightarrow 0 \end{aligned}$$

- $\mathsf{Def}(\Sigma, \vec{z}) = H^1(T\Sigma(-\log \vec{z})) = \mathbb{E}\mathsf{xt}^1(\Omega_\Sigma(\vec{z}))$
- $\mathsf{Def}(u) = H^0(u^*TX) = \mathbb{E}\mathsf{xt}^1(u^*\Omega_X \longrightarrow \Omega_\Sigma(\vec{z}), \mathcal{O}_\Sigma)$
- $\mathsf{Obs}(u) = H^1(u^*TX) = \mathbb{E}\mathsf{xt}^2(u^*\Omega_X \longrightarrow \Omega_\Sigma(\vec{z}), \mathcal{O}_\Sigma)$
- We just need $\mathsf{Obs}(f) = 0$ for $\mathcal{M}_{g,k}(X, A)$ to be smooth orbifold near f

Transversality (deformation theory): log case

Theorem (in preparation)

- For a fixed $(\Sigma, \vec{z})$ and $\mathfrak{s}$, we can construct an ∞-dimensional bundle
$$\mathcal{E}_{\log} \longrightarrow \mathcal{B}_{\log} = \mathsf{Map}_{\mathfrak{s}}((\Sigma, \vec{z}), (X, D))$$
$$\mathcal{E}_u = \Omega^{0,1}(u^*(TX(-\log D)))$$
such that $\bar{\partial} \colon \mathcal{B} \longrightarrow \mathcal{E}$ restricts to a smooth section $\bar{\partial}_{\log}$ of
$$\mathcal{E}_{\log} \longrightarrow \mathcal{B}_{\log}$$
- Zero set of $\bar{\partial}_{\log}$ is the set of log holomorphic maps of tangency type $\mathfrak{s}$ from $(\Sigma, \vec{z})$ into (X, D)
- $D_u \bar{\partial}_{\log} \colon \Omega^0(u^*TX(-\log D)) \longrightarrow \Omega^{0,1}(u^*TX(-\log D))$
-
$$\mathsf{Def}_{\log}(u) = \mathsf{ker}(D_u \bar{\partial}_{\log})$$
$$\mathsf{Obs}_{\log(u)} = \mathsf{coker}(D_u \bar{\partial}_{\log})$$

- By Riemann-Roch

$$\text{dim}_{\mathbb{R}}\text{Def}_{\log}(u) - \text{dim}_{\mathbb{R}}\text{Obs}_{\log}(u) = c_1^{TX(-\log D)}(A) + n(1-g)$$

- If $\text{Obs}_{\log}(u) = 0 \;\Rightarrow$ Around u, $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ is a smooth orbifold of real dimension

$$2\big(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k\big)$$

Theorem (in preparation) There is a class of "semi-positive" pairs (X, D) where global perturbation of $\bar{\partial}_{\log}$ (as in Ruan-Tian) allows a geometric construction of log GW invariants for (X, D)

What is left to be done?

- Constructing Virtual Fundamental Cycle (Gluing analysis)
- Comparing to algebraic log moduli spaces
- Calculating the resulting Gromov-Witten type invariants
- Looking into applications (Mirror Symmetry ...)

Thank you for your attention