Symplectic normal crossings (sub-)varieties and their smoothings

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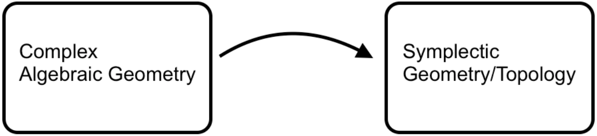
May 26 2017

Goal: bring well-known algebraic structures to symplectic topology



Pioneers: Gromov (85), Donaldson(95)

Motivations: Gromov-Witten theory, Gross-Siebert Program, Geography of symplectic manifolds

- **Symplectic manifold**:
  1. $2n$-dim manifold X, plus
  2. 2-form ω which is closed ($\mathrm{d}\omega = 0$) and non-degenerate ($\omega^n \neq 0$)

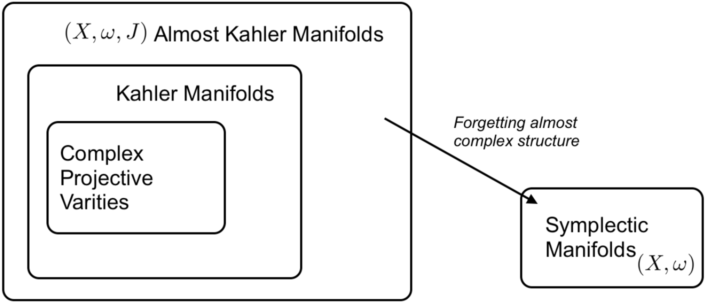
- Link to algebraic geometry: $\exists$ **almost complex structure**

$$J : TX \longrightarrow TX \qquad s.t.$$

  1. $J^2 = -\mathrm{id}$
  2. $\omega(\cdot, J\cdot)$ is a metric

- Space of such J: infinite dimensional, but **contractible**

- **Kähler** manifold: (X, ω, J) s.t. J comes from a complex str on X



- Example: $X = \mathbb{R}^{2n} = \mathbb{C}^n$, $\omega_{\mathsf{std}} = \mathsf{d}x_1 \wedge \mathsf{d}y_1 +, \ldots + \mathsf{d}x_n \wedge \mathsf{d}y_n$
- Every (X,ω) is locally isomorphic to $(\mathbb{R}^{2n}, \omega_{\mathsf{std}})$

- Algebraic geometry: studying varieties and subvarities
  1. **curves**: dim 1
  2. **divisors**: codim 1, locally $f=0$

- **Symplectic submanifold** $Y \subset X$: $\omega|_Y$ is symplectic

- Gromov(85): J-**holomorphic** curves

  - $u\colon (\Sigma, \mathrm{j}) \longrightarrow (X, J) \qquad \bar{\partial} u = \mathsf{d}u + J \circ \mathsf{d}u \circ \mathrm{j} = 0$

  - $C = \mathrm{Im}(u)$: J-holomorphic curve

  - As singular as curves in algebraic geometry

  - Moduli space of J-holomorphic maps $\rightsquigarrow$ **Gromov-Witten** theory, **Floer** homology, etc

  - $\int_C \omega > 0$: **energy bound**.

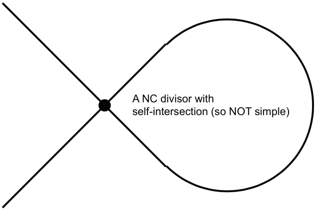
  - Particular choice of ω in its deformation equivalence class is **NOT** important

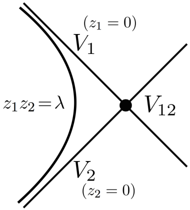
Gromov(86 Book):

He asked about **feasibility** of defining
higher dimensional singular varieties and sub-varieties
and studying their **smoothing** in symplectic category

- **Normal crossings (NC)** divisors and varieties: appear in Hodge Theory, Mirror Symmetry, compactification of moduli spaces, Gromov-Witten theory, etc.

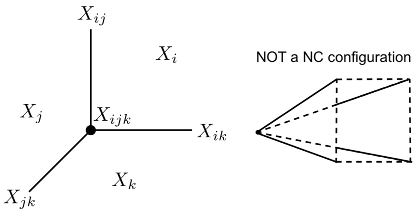
- X complex manifold $\rightsquigarrow$ an NC divisor V is locally given by $z_1 z_2 \cdots z_k = 0$ where $(z_1, \ldots, z_n)$ are local coordinates on X.





- **Simple NC divisor**: global transverse union of smooth divisors $\bigcup_{i \in S} V_i$

- **NC variety**: Locally can be embedded as simple NC divisor in $\mathbb{C}^{n+1}$
- **Simple NC variety**: Global union of smooth varieties glued along simple NC divisors
  $\bigcup_{i=1}^{N} X_i, \quad X_{ij} = X_i \cap X_j, \quad \{X_{ij}\}_{j\neq i} \subset X_i$ an SNC divisor



- **Important case of Gromov's inquiry**: Define symplectic notions of NC varieties and subvarieties and study their properties.

Some facts about smooth divisors and earlier results:

- $V \subset (X,\omega)$ smooth divisor $\rightsquigarrow$

  Normal bundle $\mathcal{N}_X V$ of V in X: rank 2 real vector bundle, inherits a symplectic form $\omega|_{\mathcal{N}_X V}$ from ω

- **Hermitian structure** $(\mathfrak{i}, \rho, \nabla)$ on $\mathcal{N}_X V$:
  - $\mathfrak{i}$: $\omega|_{\mathcal{N}_X V}$-compatible (fiber-wise) complex structure
  - ρ: Hermitian metric with real part $\rho_{\mathbb{R}} = \omega|_{\mathcal{N}_X V}(\cdot, \mathfrak{i}\cdot)$
  - ∇: connection compatible with $(\mathfrak{i}, \rho)$

- Space of such Hermitian structures: non-empty and contractible

- $\nabla \rightsquigarrow$ 1-form α on $\mathcal{N}_X V - V$ s.t. on each fiber $\mathcal{N}_X V|_x \cong \mathbb{C}$:

  $$\alpha \approx \text{1-form } \mathrm{d}\theta \quad \text{w.r.t} \quad \text{polar coordinates } (r,\theta) \text{ on } \mathbb{C}$$

- $\rho(v, \overline{v})$ on each fiber $\approx$ square norm $r^2 = |z|^2$ on $\mathbb{C}$

- $$\widehat{\omega} = \pi^*(\omega|_V) + \frac{1}{2}\mathsf{d}(\rho\alpha)$$

  - restricts to standard symplectic 2-form $\mathsf{d}(r^2\mathsf{d}\theta)$ on each fiber
  - symplectic in a small neighborhood of V in $\mathcal{N}_X V$.

- **Symplectic Neighborhood Theorem** (special case of Moser argument): There is an identification (we call it **regularization**)

$$\Psi\colon \mathsf{Dom}(\Psi) \subset \mathcal{N}_X V \longrightarrow \mathsf{Im}(\Psi) \subset X$$

such that

$$\Psi(V) = V, \qquad \mathsf{d}\Psi|_V = \mathsf{id}, \qquad \Psi^*\omega = \widehat{\omega}$$

Some applications:

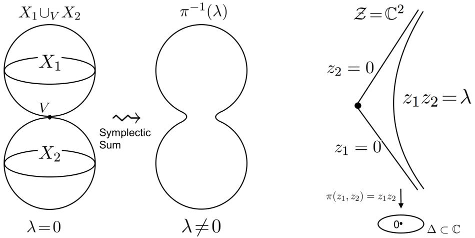
1. Space of almost complex structures J on X compatible with ω and V ($JTV = TV$): still non-empty and contractible.
2. Such $(\Psi, \mathfrak{i}) \rightsquigarrow$ a complex line bundle

$$\mathcal{O}_X(V) = \Psi_* \pi^* \mathcal{N}_X V|_{\mathsf{Im}(\Psi)} \sqcup (X - V) \times \mathbb{C}/ \sim$$
$$(\Psi(v), cv) \sim (\Psi(v), c)$$

   with $c_1(\mathcal{O}_X(V)) = \mathsf{PD}(V)$ as in algebraic geometry.

3. Symplectic Sum Construction Gompf (94), McCarthy-Wolfson (94): First realization of Gromov's inquiry to define singular varieties and study their smoothings in symplectic categor

- (X_1, ω_1) and (X_2, ω_2) with smooth symplectic divisor $V \subset X_1, X_2 \rightsquigarrow$ simple NC variety $X_1 \cup_V X_2$ with two components.



- **Question**: Does there exists

$$\begin{array}{ccc} X_1 \cup_V X_2 & \longrightarrow & (\mathcal{Z}, \omega) \\ \downarrow \pi & & \downarrow \pi \\ 0 & \longrightarrow & \Delta \subset \mathbb{C} \end{array}$$

and all other fibers are smooth symplectic submanifolds.

- **Symplectic Sum Theorem**: $X_1 \cup_V X_2$ is smoothable $\Leftrightarrow \mathcal{N}_{X_1}V$ and $\mathcal{N}_{X_2}V$ are dual to each other, i.e.

$$c_1(\mathcal{N}_{X_1}V) + c_1(\mathcal{N}_{X_2}V) = 0 \quad \text{or} \quad \mathcal{N}_{X_1}V \otimes \mathcal{N}_{X_2}V \cong V \times \mathbb{C}$$

- The construction of $\pi\colon \mathcal{Z} \longrightarrow \Delta$ uses
  1. regularizations $\Psi_i\colon \mathrm{Dom}(\Psi_i) \subset \mathcal{N}_{X_i}V \longrightarrow \mathrm{Im}(\Psi_i) \subset X$, for $i = 1, 2$,
  2. a trivialization $\Phi\colon \mathcal{N}_{X_1}V \otimes \mathcal{N}_{X_2}V \longrightarrow V \times \mathbb{C}$
  3. various cut-off functions.

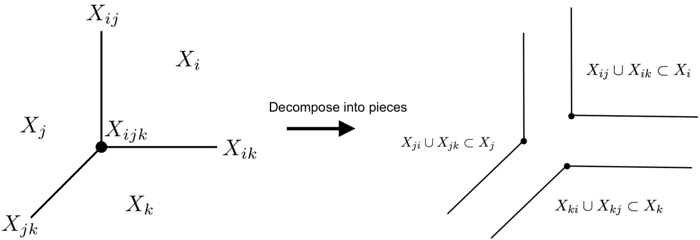
- **Conclusion**:
  Since only deformation equivalence class of smooth fibers of $\mathcal{Z}$ (as the out put of the construction) is well-defined $\rightsquigarrow$
  Only the deformation equivalence class of (X_1, ω_1), (X_2, ω_2), Φ (as inputs of this construction) matter and the particular choice is not important.

A cool application of the symplectic sum construction in Gompf's paper:

Theorem(Gompf 94): Every finitely generated group G can be realized as the fundamental group of a smooth closed symplectic manifold of real dimension 4.

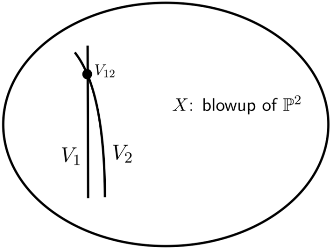
Questions

1. How to define NC symplectic varieties with (locally) more than two components?



2. How to define NC symplectic divisors?
3. Which NC symplectic varieties are smoothable?

- **First Guess**: Transverse union $\{V_i\}_{i\in S}$ of smooth divisors
- **Symplectic-ness issue**: The intersections $V_I = \bigcap_{i\in I} V_i$ do not have to be symplectic
- **Orientation issue**, example: X: blowup of $\mathbb{P}^2$ at a point, V_1: exceptional divisor, V_2: a C^1-small deformation of that s.t. $V_{12} = V_1 \cap V_2$ is a point with intersection number -1



- **simple NC symplectic divisor:** A transverse union of smooth symplectic divisors $V \equiv \{V_i\}_{i \in S}$ in (X, ω) s.t.
  1. all V_I are symplectic
  2. **intersection orientation** $=$ **symplectic orientation** for each V_I

- Symp(X, V) $=$ space of all symplectic structures ω such that V is an NC symplectic divisor in (X, ω)

- Blow-up example above: $V_{12} = V_1 \cap V_2$ is a point with the intersection orientation -1.
  However, the symplectic orientation of any point is $\omega^0 = +1$

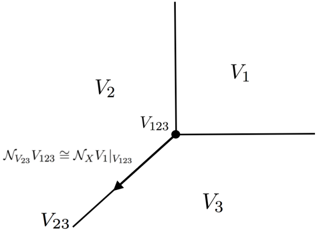
- $X = \mathbb{C}^3$, $V_i = (z_i = 0) \cong \mathbb{C}^2$ (coordinate hyperplanes) for $i = 1, 2, 3$.

$$\begin{aligned}\omega = \; & dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 \\ & -a(dx_1 \wedge dy_2 + dx_2 \wedge dy_1 + dx_1 \wedge dy_3 + dx_3 \wedge dy_1)\end{aligned}$$

- V_I are all symplectic if $a \neq \pm\sqrt{.5}, \pm 1$
- $|a| > 1 \rightsquigarrow$ Not satisfied along V_{23}
- $\sqrt{.5} < |a| < 1 \rightsquigarrow$ Not satisfied along V_{12}, V_{13}, V_{23}
- $|a| < \sqrt{.5} \rightsquigarrow$ ✓

What is the analogue of Symplectic Neighborhood Theorem for NC symplectic divisors?

- For every $I \subset S$, $\mathcal{N}_X V_I \cong \bigoplus_{i \in I} \mathcal{N}_{V_{I-i}} V_I \cong \bigoplus_{i \in I} \mathcal{N}_X V_i|_{V_I}$



- $\forall\ j \in I$, fix Hermitian structure $(\mathfrak{j}_j, \rho_{I;j}, \nabla_{I;j})$ on $\mathcal{N}_{V_{I-j}} V_I$
- The closed 2-form

$$\widehat{\omega}_I = \pi^*(\omega|_{V_I}) + \frac{1}{2} \sum_{j \in I} \mathrm{d}(\rho_{I;j} \alpha_{I;j})$$

is a symplectic structure on a neighborhood of V_I in $\mathcal{N}_X V_I$.

An ω-**regularization** for $V \equiv \{V_i\}_{i \in S}$ in (X, ω) is a set of identifications

$$\mathcal{R} = \{\Psi_I \colon \mathrm{Dom}(\Psi_I) \subset \mathcal{N}_X V_I \longrightarrow \mathrm{Im}(\Psi_I) \subset X\} \qquad \text{such that}$$

- stratified: $\Psi_I(\mathrm{Dom}(\Psi_I) \cap \mathcal{N}_{V_{I'}} V_I) = \mathrm{Im}(\Psi_I) \cap V_{I'}$.
- identity to the first order: $\mathrm{d}\Psi_I|_{V_I} = \mathrm{id}$
- symplectic: $\Psi_I^* \omega = \widehat{\omega}_I$
- Ψ_I are compatible in a suitable sense.

- **Question**: Does every NC symplectic divisor admit a regularization? **NO**, except in the smooth case!!!!!!!
- **Why?** A necessary condition for existence of a regularization or an almost complex structure which is compatible with all V_i is that they intersect **orthogonally**.
- $\mathrm{Aux}(X, V)$ = space of all pairs $(\mathcal{R}, \omega)$ such that $\omega \in \mathrm{Symp}(X, V)$ and $\mathcal{R}$ is an ω-regularization for $\{V_i\}_{i \in S}$ in X.

- **Theorem**(F.T., McLean, Zinger): The map

$$\mathsf{Aux}(X, \{V_i\}_{i \in S}) \longrightarrow \mathsf{Symp}(X, \{V_i\}_{i \in S}), \quad (\mathcal{R}, \omega) \longrightarrow \omega$$

is a weak homotopy equivalence (i.e. equal π_i for all $i \geq 0$).

- What does it mean in practice?
  1. Any ω in Symp(X, V) can be deformed in a small neighborhood of the singular locus to a cohomologous ω' such that $\{V_i\}_{i \in S}$ admits an ω'-regularization.
  2. Given $(\mathcal{R}_0, \omega_0)$ and $(\mathcal{R}_1, \omega_1)$, and a path of symplectic forms $(\omega_t)_{t \in [0,1]}$ in Symp(X, V) connecting ω_0 and ω_1, there exists a cohomologous deformation of this path with the same end points $(\omega'_t)_{t \in [0,1]}$ that lifts to $(\mathcal{R}'_t, \omega'_t)$ with

$$\mathcal{R}'_0 \approx \mathcal{R}_0 \qquad \mathcal{R}'_1 \approx \mathcal{R}_1.$$

Moral conclusion: Our definition of NC symplectic divisor is good for any application that mostly cares about **deformation equivalence class** of symplectic structures, such as Gromov-Witten theory, symplectic sum construction, etc.

Some applications:

(1) Space of good almost Kähler structures:

$\mathsf{AK}(X,V)$: Space of triples $(J,\mathcal{R},\omega)$ s.t. $(\mathcal{R},\omega)\in\mathsf{Aux}(X,V)$ and J is a compatible with $(\mathcal{R},\omega)$ in the sense that

1. $\omega(\cdot,J\cdot)$ is a metric
2. $JTV_i=TV_i$ for all $i\in S$
3. $\Psi_I^*J=\pi^*(J|_{V_I})\oplus\bigoplus_{j\in I}\pi^*\mathfrak{i}_{I;j}$ for all $I\subset S$

- **Proposition**(F.T., McLean, Zinger) The map

$$\mathsf{AK}(X,V)\longrightarrow\mathsf{Symp}(X,V),\quad (J,\mathcal{R},\omega)\longrightarrow\omega$$

  is also a weak homotopy equivalence.

- **Conclusion** We can use moduli space of J-holomorphic maps with J in $\mathsf{AK}(X,V)$ to define invariants of deformation equivalence classes of symplectic structures on $(X,\{V_i\}_{i\in S})$

(2) Natural vector bundles associated to NC divisors:

- A pair $(\mathcal{R}, \omega) \rightsquigarrow$ a complex line bundle $\mathcal{O}_X(V)$ with $c_1(\mathcal{O}_X(V)) = \mathsf{PD}(V)$ whose deformation equivalence class only depends on the class of $\omega \in \mathsf{Symp}(X, V)$
- A triple $(J, \mathcal{R}, \omega) \rightsquigarrow$ a complex vector bundle

$$TX(-\log V) \longrightarrow X$$

which generalizes the notion of **logarithmic tangent bundle** in algebraic geometry. Deformation equivalence class of $TX(-\log V)$ only depends on the class of $\omega \in \mathsf{Symp}(X, V)$.

(3) Smoothing of NC varieties (generalization of symplectic sum):

- Simple NC symplectic variety $X_\emptyset = \bigcup_{i=1}^{N} X_i$, with singular locus

$$X_\partial = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{N} X_{ij}$$

  we can define a complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset) \longrightarrow X_\partial$ whose restriction to each X_{ij} is equal to

$$\mathcal{N}_{X_i} X_{ij} \otimes \mathcal{N}_{X_j} X_{ij} \otimes \bigotimes_{k \neq i,j} \mathcal{O}_{X_{ij}}(X_{ijk})$$

- Friedman (1983): In complex algebraic geometry, triviality of $\mathcal{O}_{X_\partial}(X_\emptyset)$ (d-semistability condition) is a necessary condition for the smoothability of $X_\emptyset$ in a smooth one-parameter family. The condition is however not sufficient even if $N=2$ (U. Persson and H. Pinkham)

- **Theorem** (F.T., McLean, Zinger) A NC symplectic variety is smoothable (in the symplectic category) if and only if $\mathcal{O}_{X_\partial}(X_\emptyset)$ is isomorphic to the trivial complex line bundle.

- The proof is constructive
- Depends on a regularization $(\mathcal{R},\omega)\in\mathsf{Aux}(X_\emptyset)$, a compatible trivialization of $\mathcal{O}_{X_\partial}(X_\emptyset)$, and lots of other auxiliary smooth functions
- Smooth fibers of the smoothing are called **multifold** symplectic sum of $X_\emptyset$
- The deformation equivalence class of the multifold symplectic sum only depends on the deformation equivalence class of the symplectic structure on $X_\emptyset$ and the homotopy class of the trivialization of $\mathcal{O}_{X_\partial}(X_\emptyset)$.

Some words on proofs

(1) The essential idea in the proof of regularization theorem:

- Consider $\mathbb{C}^N$ with $\omega_{\mathsf{std}} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots, \mathrm{d}x_n \wedge \mathrm{d}y_n$ and coordinate hyperplanes $V_i \equiv (x_i = y_i = 0)$.
- **Lemma**: Let ω be a linear symplectic form on $\mathbb{C}^N$ such that ω-orientation and complex orientation (intersection orientation) of every V_I are the same. Then ω can be deformed (via linear symplectic forms of the same sort) to ω_{std}.
- This lemma exhibits the necessity of orientation condition in the definition of SC divisors.

Proof of Lemma

- $\omega^\bullet = \oplus\omega|_{\mathbb{C}^i}$: this is diagonal part of ω
- $\omega^\circ = \omega - \omega^\bullet$
- We increase diagonal part of ω by replacing ω with $\omega_t = \omega + t\omega^\bullet$, $t >> 0$
- We eliminate the off diagonal part
- We scale it back to $\omega^\bullet$
- Note that $\omega^\bullet$ is deformable to ω_{std}
- In practice we need to consider vector spaces $\mathcal{N}_{V_I}X$ with $\omega = \omega_V + \frac{1}{2}\mathsf{d}(\iota_\zeta\Omega_\nabla)$ instead of just $\mathbb{C}^N$ which makes the argument technical, specially if V_I is not compact, and inductively build and merge these deformations.

(2) necessity of d-semistability condition:

- Extension Lemma: $\pi\colon L \longrightarrow M$ a Hermitian line bundle, and $M' \subset M$ complement of closed submanifolds $V_1, \ldots, V_\ell \subset M$ of real codimension c or higher. If $c \geq 2$, any two trivializations of L over M that restrict to homotopic trivializations of $L|_{M'}$ are homotopic as trivializations of L over M. If $c \geq 3$, every trivialization of $L|_{M'}$ is homotopic to the restriction of a trivialization of L over M.
- **Proposition**: Let $\pi\colon \mathcal{Z} \longrightarrow \Delta$ be a one-parameter family of smoothings of a connected simple NC symplectic variety $X_\emptyset = \bigcup_{i=1}^N X_i$, and $s_\emptyset$ be the canonical section of the line bundle

$$\mathcal{O}_{\mathcal{Z}}(X_\emptyset) = \bigotimes_{i=1}^{N} \mathcal{O}_{\mathcal{Z}}(X_i)$$

Then there exists a non-vanishing section s of $\mathcal{O}_{\mathcal{Z}}(X_\emptyset)$ so that the smooth maps $s_\emptyset/s|_{\mathcal{Z}-X_\emptyset}$ and $\pi|_{\mathcal{Z}-X_\emptyset}$ to $\mathbb{C}^*$ are homotopic.

Tak for din opmærksomhed