1. Let $B_1$ be the unit disk in $\mathbb{R}^2$. Show that the following problem has a unique bounded solution of:

$$
\begin{cases}
\Delta u = 0 \text{ in } B_1 \\
u = 1 \text{ on } \partial B_1^+ \\
u = -1 \text{ on } \partial B_1^-
\end{cases}
$$

Discuss $\lim u(x,y)$ along rays to $(\pm 1,0)$.

2. Let $B_1$ be the unit ball in $\mathbb{R}^n$. Prove that for any solution $u$ of

$$
\begin{cases}
\Delta u = f \text{ in } B_1 \\
u = 0 \text{ on } \partial B_1
\end{cases}
$$

Then (a) $\|u\|_{L^\infty(B_1)} \leq \frac{1}{2^n}\|f\|_{L^\infty(B_1)}$.
(b) $\|u\|_{L^2(B_1)} \leq C\|f\|_{L^2(B_1)}$,

for some constant $C$.

Identify the best constant $C$.

Moreover show that there is no constant $C$ so that for any smooth function $u$ with

$$
\begin{cases}
\Delta u = f \text{ in } B_1 \\
u = 0 \text{ on } \partial B_1
\end{cases}
$$

implies $|u(x)| \leq C|f(x)|$ for all $x$ in $B_1$.

3. Show that there is a unique radially symmetric solution of

$$
\begin{cases}
-\Delta u = 1 - sin(u^2) \text{ in } B_1 \\
u = 0 \text{ on } \partial B_1
\end{cases}
$$

4. Let $u(x,t)$ be the weak solution of

$$
\begin{cases}
u_t - \Delta u = 0 \text{ in } B_1 \times (0, \infty), \\
u = f(x) \text{ on } B_1 \times \{0\} \\
u = 0 \text{ on } \partial B_1 \times (0,1].
\end{cases}
$$

Show that for some universal constant $C$,

$$
\int_{B_1 \times [0,1]} u^2 + |\nabla u|^2 \leq C \int_{B_1} f^2
$$
and that
\[ |u(x, t)| \leq C\varphi_1(x)e^{-\lambda_1 t}\|f\|_{L^2(B_1)} \]
for \( t \geq 1 \). Here \( \varphi_1 \) is the first eigenfunction of \(-\Delta\) with eigenvalue \( \lambda_1 > 0 \). Is the above inequality true for \( t \geq 0 \)?

5. Let \( \Omega \) be a bounded smooth domain in \( R^2 \). Show that there are at least three solutions of
\[
\begin{cases}
-\Delta u = u^3 \text{ in } \Omega \\
u = 0 \text{ on } \partial\Omega.
\end{cases}
\]

6. Show that for each smooth function \( f(x) \), there is a solution of
\[
\begin{cases}
div((1 + \chi_{B_1})\nabla u) = 0 \text{ in } B_2 \\
u = f(x) \text{ on } \partial B_2.
\end{cases}
\]
Here
\[
\chi_{B_1}(x) = \begin{cases}
1 & \text{if } x \text{ is in } B_1 \\
0 & \text{else}.
\end{cases}
\]
Find how \( \nabla u \) jumps along \( \partial B_1 \).