1. Find $\frac{dy}{dx}$ for the following function

$$y = \ln(\cos x).$$

Solution:

$$\frac{dy}{dx} = \frac{1}{\cos x} (\cos x)' = \frac{-\sin x}{\cos x} = -\tan x.$$

2. Find $\frac{dy}{dx}$ for the following function

$$y = e^{\cos x}.$$

Solution:

$$\frac{dy}{dx} = e^{\cos x} (\cos x)' = -(\sin x)e^{\cos x}.$$

3. Find $\frac{dy}{dx}$ for the following function

$$y = \ln(\sin(2x)).$$

Solution:

$$\frac{dy}{dx} = \frac{1}{\sin 2x} (\sin 2x)' = \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x.$$

4. Find $\frac{dy}{dx}$ for the following function

$$y = e^x \cos x.$$

Solution (product rule!):

$$\frac{dy}{dx} = e^x (\cos x) + e^x (-\sin x) = e^x (\cos x - \sin x).$$

5. Find $\frac{dy}{dx}$ for the following function

$$y = e^{-x} \sin x.$$

Solution (product rule again!):

$$\frac{dy}{dx} = -e^{-x} (\sin x) + e^{-x} (\cos x) = e^{-x} (-\sin x + \cos x) = e^{-x} (\cos x - \sin x).$$
6. Find $\frac{dy}{dx}$ for the following function

$$y = \int_{-\infty}^{x} e^{-t^2} dt.$$ 

NOTE: Before you try this, you should be familiar with Theorem 7.3.2 on page 327!!
(Often referred to as one version of the Fundamental Theorem of Calculus!!)
Solution: Once you read the theorem, you will see the answer is simply:

$$e^{-x^2}.$$ 

7. Find $\frac{dy}{dx}$ for the following function

$$y = \int_{x}^{\infty} e^{-t^2} dt.$$ 

Solution: We are again going to use the above theorem, but careful to note that the variable x is on the bottom of the integral now... This time:

$$y = \int_{x}^{\infty} e^{-t^2} dt = -\int_{\infty}^{x} e^{-t^2} dt$$

So now we can use that theorem and get the answer:

$$-e^{-x^2}.$$ 

8. Find $\frac{dy}{dt}$ for the following function

$$y = \int_{t}^{2} (1 + x)^3 dx.$$ 

Solution: As in #7, we have to note where the variable is before using the theorem as it is stated. So:

$$y = \int_{t}^{2} (1 + x)^3 dx = -\int_{2}^{t} (1 + x)^3 dx,$$

and hence we can use the theorem to get our answer:

$$-(1 + t)^3.$$ 

9. Calculate $dy/dx$ for the following pair of parametric equations

$$\begin{cases} x &= \sin 2\theta \\ y &= 3 + \cos \theta. \end{cases}$$
Solution:
\[
\frac{dx}{d\theta} = 2 \cos(2\theta) \\
\frac{dy}{d\theta} = -\sin \theta
\]

and so
\[
\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \left(\frac{d\theta}{dx}\right) = \frac{-\sin \theta}{2 \cos(2\theta)}
\]

10. Determine the period and amplitude of the following function
\[
y = 3 \sin 4t + 4 \cos 4t.
\]
Solution: The period is \(\frac{2\pi}{4} = \frac{\pi}{2}\) and the amplitude is \(\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5\).

11. Determine the period and amplitude of the following function
\[
y = 3 \sin 5t - 4 \cos 5t.
\]
Solution: The period is \(\frac{2\pi}{5}\) and the amplitude is \(\sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5\).

12. Determine the period and amplitude of the following function
\[
y = -\sin 4x + 4 \cos 4x.
\]
Solution: The period is \(\frac{2\pi}{4} = \frac{\pi}{2}\) and the amplitude is \(\sqrt{(-1)^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}\).

13. Determine the period and amplitude of the following function
\[
y = 3 \sin(4t + 1) + 4 \cos(4t + 1).
\]
Solution: The period is \(\frac{2\pi}{4} = \frac{\pi}{2}\) and the amplitude is \(\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5\).

14. Approximate \(\sqrt{15}\) by tangent line approximation using \(\sqrt{16} = 4\).

15. Approximate \(\sqrt[3]{29}\) by tangent line approximation using \(\sqrt[3]{27} = 3\).

16. Approximate \(\frac{1}{\sqrt{16}}\) by tangent line approximation using \(\frac{1}{\sqrt{16}} = \frac{1}{4}\).

The problems 14-16 are based on the tangent approximation
\[
(1 + x)^\alpha \approx 1 + \alpha x
\]

where \(1 + \alpha x\) is the tangent line at \(x = 0\).
This is a good approximation only when $x$ is small. In practice, you have to be able to use it in creative ways.

For Problem 14, it is clear that 15 is close to 16 which is a perfect square. Hence

$$\sqrt{15} = \sqrt{16-1} = \sqrt{16(1 - \frac{1}{16})} = 4\sqrt{1 - \frac{1}{16}} \approx 4(1 - \frac{1}{32}) = 4 - \frac{1}{8} = \frac{31}{8} = 3.875.$$ 

One can also compute this by taking $f(x) = \sqrt{x}$ and $x_0 = 16, x = 15$. Then compute the tangent line at $x_0$. The answer would be exactly the same since the uniqueness of tangent line.

For 15, one write

$$\sqrt{29} = \sqrt{27+2} = \sqrt{27(1 + \frac{2}{27})} = 3\sqrt{1 + \frac{2}{27}} \approx 3(1 + \frac{2}{81}) = 3 + \frac{2}{27} = \frac{83}{27}.$$ 

Similar for Problem 16, one compute as

$$\frac{1}{\sqrt{15}} = \frac{1}{\sqrt{16-1}} = \frac{1}{4\sqrt{1 - \frac{1}{16}}} = \frac{1}{4}(1 - \frac{1}{16})^{-\frac{1}{2}} \approx \frac{1}{4}(1 - \frac{1}{2} \cdot (-\frac{1}{16})) = \frac{1}{4}(1 + \frac{1}{32}) = \frac{33}{128}.$$ 

17. Find the inflection points of the function $e^{-x^2}$. Inflection points are the points the function changes its convexity. Therefore, we have to find where its second order derivative changes its sign, which happens usually, but not always, at its zeros.

$$(e^{-x^2})' = e^{-x^2}(-2x), (e^{-x^2})'' = e^{-x^2}(4x^2 - 2) = 0.$$ 

It has two zeros $\pm \frac{1}{\sqrt{2}}$ and $(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$ are the inflection points.

18. Find the inflection points of the function $xe^{-x^2}$. As the last problem,

$$(xe^{-x^2})' = e^{-x^2}(-2x^2 + 1), (e^{-x^2})'' = e^{-x^2}(4x^3 - 2x - 4x) = e^{-x^2}x(4x^2 - 6).$$ 

Hence the inflection points are at 0 and at $\pm \frac{\sqrt{3}}{\sqrt{2}}$. Finally the inflection points are $(0,0)$ and $\pm (\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}e^{-\frac{3}{2}})$.

19. What is the maximum value of the function $y = x^2 - 2x$ in the interval $[0,3]$.

One important warning for this kind of problems is that the extreme value might happen on the boundary.

There are two steps: Compute zeros the derivate: $2x - 2 = 0$. We got $x = 1$.

Then we compare: at $x = 1, y = -1$. Since the boundary of the interval is 0 and 3. We compute those values: At $x = 0, y = 0$ and at $x = 3, y = 3^2 - 2 * 3 = 3$. Hence 3 is the maximum value.
20. What is the minimum value of the function \( y = x^2 - 2x \) in the interval \([2, 3]\)?

We follow as in the previous problem. One zero of the derivative is at \( x = 1 \). Since it is outside the interval of consideration, we don’t have put it into comparison. Then we compute at \( x = 2 \), \( y = 2^2 - 2 \times 2 = 0 \) and at \( x = 3 \), \( y = 3^2 - 2 \times 3 = 3 \). Hence 0 is the minimum value. (3 would be maximum value again!)

21. Suppose the cost the top of a box with square base is twice as expensive as the bottom and the sides. Suppose also the volume of the box is 2 feet\(^3\). What is the height of the box that can be build with the lowest possible cost?

Let \( b \) be the length of the side of square base and \( h \) be the height. The the volume has to be \( b^2h = 2 \). The cost, called it \( c \) should be

\[
c(b) = 4bh + b^2 + 2b^2,
\]

where we have assume the the unit price for the sides and bottom is 1. Plugging in \( h = \frac{2}{b^2} \). We have

\[
c = \frac{8}{b} + 3b^2,
\]

which should be considered as a function defined on all positive \( b \). Now take derivative in \( b \) of the above function:

\[
-\frac{8}{b^2} + 6b,
\]

so its zero is \( b = \frac{\sqrt[3]{4}}{\sqrt{3}} \). (The negative solution isn’t physical.) Since the cost at the boundary 0 and \( \infty \) is clearly \( \infty \), we have the minimum cost is at \( b = \frac{\sqrt[3]{4}}{\sqrt{3}} \). And the height is \( \frac{2\sqrt[3]{4}}{\sqrt{16}} \).

22. What is the maximum value of the function \( y = \frac{1}{3}x^3 + x^2 - 3x \) in the interval \([1, 3]\).

As before we compute the derivative:

\[
y' = x^2 + 2x - 3 = 0
\]

and obtain the zeros as \( x = -3 \) and \( x = 1 \). We should omit \(-3\) since it is outside the interval of consideration. Then we compare:

\[
x = 1, y = \frac{1}{3} + 1 - 3; x = 3, y = 9 + 9 - 6.
\]

So the maximum is at 3 with maximum value 12.

23. What is the maximum value of the function \( y = x^2 - 2x \) in the interval \([1, 3]\).

As before, the zero of the derivative is 1 which is one the boundary. Hence we only have to compare the values at the boundary points: \( x = 1, y = -1 \), and at \( x = 3, y = 3 \). Hence 3 again is the maximum.
24. What is the minimum value of $x^2 + y^2$ if $x + y = 1$?

Since $x + y = 1$, $y = 1 - x$, so we are minimizing $f(x) = x^2 + (1 - x)^2$, then $\frac{df}{dx} = 2x - 2(1 - x)$. Equating this to zero we get

$$0 = 2x - 2 + 2x$$

$$2 = 4x$$

$$x = \frac{1}{2}$$

Thus, if the minimum exists it occurs at $x = \frac{1}{2}$, so that $y = \frac{1}{2}$ and the minimum value of $f$ is $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}^2 + \frac{1}{2}^2 = \frac{1}{2}$.

25. What is the maximum value of $x + y$ if $x^2 + y^2 = 5$?

As before $f(x, y) = x + y$. If $x^2 + y^2 = 5$, then

$$x^2 - 5 = -y^2$$

$$y = \pm \sqrt{5 - x^2}$$

so we are minimizing

$$x \pm \sqrt{5 - x^2}$$

$$\frac{df}{dx} = 1 + \frac{1}{2}(5 - x^2)^{-1/2}(-2x)$$

equate to zero

$$0 = 1 \pm \frac{x}{\sqrt{5 - 2}}$$

$$-1 = \pm \frac{x}{\sqrt{5 - 2}}$$

$$-\sqrt{5 - 2} = \pm x$$

square both sides and get

$$\sqrt{5 - 2} = x$$

$$5 - x^2 = x^2$$

$$5 = 2x^2$$

$$\frac{5}{2} = x^2$$

$$\sqrt{\frac{5}{2}} = x$$

substituting this back into the constraint we get that

$$\sqrt{\frac{5}{2}} + y^2 = 5$$
\[
\frac{5}{2} + y^2 = 5
\]
\[
y^2 = \frac{5}{2}
\]
\[
y = \sqrt{\frac{5}{2}}
\]
so that the minimum value of \( f \) is
\[
f(\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}) = \sqrt{\frac{5}{2}} + \sqrt{\frac{5}{2}} = 2\sqrt{\frac{5}{2}}
\]

26. What is the minimum value of \( x^3 + y^2 \) if \( x + y = 1 \) and both \( x, y \) are positive?

\[
f(x, y) = x^3 + y^2
\]
, since \( x + y = 1, y = 1 - x \) so we are minimizing

\[
x^3 + (1 - x)^2 = x^3 + x^2 - 2x + 1
\]
\[
\frac{df}{dx} = 3x^2 + 2x - 2
\]
equating with zero we get

\[
0 = 3x^2 - 2
\]
Solve using quadratic formula to obtain

\[
x = -\frac{1}{3} \pm \frac{1}{3}\sqrt{7}
\]
. Since both \( x \) and \( y \) are positive \( x = -\frac{1}{3} + \frac{1}{3}\sqrt{7} \) so \( y = 1 - (-\frac{1}{3} + \frac{1}{3}\sqrt{7}) = \frac{4}{3} - \frac{1}{3}\sqrt{7} \)
Thus, the min of \( f \) is \( f(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, \frac{4}{3} - \frac{1}{3}\sqrt{7}) = (-\frac{1}{3} + \frac{1}{3}\sqrt{7})^3 + (\frac{4}{3} - \frac{1}{3}\sqrt{7})^2 \).

27. What is the maximum value of \( x + 2y \) if \( x^2 + y^2 = 5 \)? We solve the \( y = \pm\sqrt{5 - x^2} \.
Notice this has domain of definition \([-\sqrt{5}, \sqrt{5}] \). Hence the function becomes

\[
f(x) = x \pm 2\sqrt{5 - x^2}
\]
Now we take derivative in \( x \) and compute the zeros as

\[
1 \pm \frac{-2x}{\sqrt{5 - x^2}} = 0.
\]
\[
\sqrt{5 - x^2} = \pm 2x.
\]
\[
5 - x^2 = 4x^2.
\]
And \( x = \pm 1 \). (Notice this \( \pm \) has nothing to do with the previous \( \pm \)) Now we compare the values: At \( \pm 1 \), the function \( f(\pm 1) = \pm 1 \pm 4 \). At \( \pm \sqrt{5}, f = 1 \). Hence the maximum value of \( 1 + 5 = 6 \) and the minimum is \( -1 - 4 = -5 \).
28. What is the tangent line of the ellipse function
\[
\frac{1}{4}x^2 + \frac{1}{6}y^2 = 1,
\]
at the point \((\sqrt{2}, \sqrt{3})\).

Find derivative implicitly to get
\[
\frac{1}{2}x + \frac{1}{3}y \frac{dy}{dx} = 0
\]

Substitute in \(x = \sqrt{2}, y = \sqrt{3}\) and get
\[
\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} \frac{dy}{dx} = 0
\]
\[
\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \frac{dy}{dx} = 0
\]
\[
\frac{1}{\sqrt{3}} \frac{dy}{dx} = -\frac{1}{\sqrt{2}}
\]
\[
\frac{dy}{dx} = -\frac{\sqrt{3}}{\sqrt{2}}
\]

Hence the tangent line is
\[
y - \sqrt{3} = -\frac{\sqrt{3}}{\sqrt{2}} (x - \sqrt{2}),
\]
which is also
\[
y = -\frac{\sqrt{3}}{\sqrt{2}} x + 2\sqrt{3}.
\]

29. Which is the tangent line of the ellipse function
\[
\frac{1}{8}x^2 + \frac{1}{4}y^2 = 1,
\]
at the point \((-2, \sqrt{3})\).

Find the derivative \(\frac{dy}{dx}\), we get
\[
\frac{1}{4}x + \frac{1}{2}y \frac{dy}{dx} = 0
\]
\[
\frac{1}{2}y \frac{dy}{dx} = -\frac{1}{4}x
\]
\[
\frac{dy}{dx} = -\frac{x}{2y}
\]

So at the point \((-2, \sqrt{3})\) this is \(\frac{dy}{dx} = \frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}\)
30. Which is the tangent line of the implicitly defined curve
\[ \frac{1}{4}x^2 + \frac{3}{4}y^2 = 2 + x - 2y, \]
at the point \((-2, -2)\).
Find \(\frac{dy}{dx}\) implicitly and get
\[ \frac{1}{2}x + \frac{6}{4}y \frac{dy}{dx} = 1 - 2 \frac{dy}{dx} \]
substitute \(x = -2, y = -2\) to get that
\[ -1 - 3 \frac{dy}{dx} = 1 - 2 \frac{dy}{dx} \]
so that
\[ -2 = \frac{dy}{dx} \]
Therefore the tangent line is
\[ y + 2 = -2(x + 2), \text{i.e., } y = -2x - 6. \]

31. What is the area of the region bounded by the curve \(y = x^2\), \(x\)-axis and the line \(x = 1\)?
This is asking for
\[ \int_{0}^{1} x^2 dx = \frac{x^3}{3} \bigg|_{0}^{1} = \frac{1}{3} - 0 = \frac{1}{3} \]
Therefore the tangent line is
\[ y - \sqrt{3} = \frac{1}{3}(x + 2). \]

32. Compute
\[ \int_{0}^{2} \frac{1}{4 + x^2} dx. \]
From the integration table we know that the antiderivative of \(\frac{1}{a^2 + x^2}\) is \(\frac{1}{a}Tan^{-1}\left(\frac{x}{a}\right)\) so that this becomes
\[ \frac{1}{2}Tan^{-1} \left(\frac{x}{2}\right) \bigg|_{0}^{2} = \frac{1}{2}(Tan^{-1}(1) - Tan^{-1}(0)) = \frac{\pi}{8}. \]

33. Compute
\[ \int_{0}^{2} \frac{x}{9 + x^2} dx. \]
First we compute the antiderivative.
Let \(u = 9 + x^2\) then
\[ \frac{du}{dx} = 2x, \ dx = \frac{du}{2x} \]
\[ \int \frac{x}{9 + x^2} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln(u) = \frac{1}{2} \ln(9 + x^2). \]

Hence
\[ \int_0^2 \frac{x}{9 + x^2} \, dx = \left[ \frac{1}{2} \ln(9 + x^2) \right]_0^2 = \frac{1}{2} (\ln 13 - \ln 9). \]

34. (This kind of problems won’t be on the test) Find the approximation \( A_n \) to the curve \( y = 6 - 2x \) bounded by the lines \( x = 0 \) and \( x = 2 \) by dividing the area into 4 rectangles. (You may want to draw a picture first!)

Divide the line \([0,2]\) into 4 equal components so that \( x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2 \) then
\[
\sum_{k=1}^{4} x_k \cdot f(x_k) = \frac{1}{2} \cdot f(\frac{1}{2}) + \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f(\frac{3}{2}) + \frac{1}{2} \cdot f(2)
\]
\[
= \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 2
\]
\[
= 7
\]

35. Evaluate the following definite integral:
\[ \int_1^2 (4x^3 - 5x^2 + 6) \, dx \]

use the simple antiderivative to find that this is
\[
x^4 - \frac{5}{3}x^3 + 6x \bigg|_1^2
\]
\[
= (16 - \frac{5}{3}8 + 12) - (1 - \frac{5}{3} + 6)
\]

36. Evaluate the following definite integral \( \int_{-2}^{4} (5t - 3e^t) \, dt \) this becomes
\[
\frac{5}{2}t^2 - 3e^t \bigg|_{-2}^{4} = \left( \frac{5}{2}16 - 3e^4 \right) - \left( \frac{5}{2}4 - 3e^{-2} \right) = (40 - 3e^4) - (10 - 3e^{-2})
\]

37. Evaluate the following definite integral \( \int_0^2 (y^2 \cos y^3) \, dy \)

We will proceed by doing a substitution. Let
\[ u = y^3 \]
then
\[ du = 3y^2 \, dy \]
or
\[ \frac{1}{3} du = y^2 dy \]

So we have
\[ \int_{0}^{2} (y^2 \cos y^3) dy = \int_{0}^{8} \cos u \frac{1}{3} du \]

notice we had to change the limits of integration since for \( y \) between 0 and 2 we have that \( u \) is between 0 and 8. Now we may compute the integral
\[ \frac{1}{3} \int_{0}^{8} \cos u du = \frac{1}{3} \sin 8 - \frac{1}{3} \sin 0 = \frac{1}{3} \sin 8. \]

38. Compute
\[ \int x \sqrt{1 + x^2} dx. \]

Let \( u = 1 + x^2 \) then \( \frac{1}{2} du = x dx \). Then
\[ \int x \sqrt{1 + x^2} dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{2} \frac{2}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (1 + x^2)^{\frac{3}{2}} + C \]

39. Compute
\[ \int x \sqrt{1 - x^2} dx. \]

Let \( u = 1 - x^2 \) then \( -\frac{1}{2} du = x dx \). So we have
\[ \int x \sqrt{1 - x^2} dx = -\frac{1}{2} \int \sqrt{u} du = -\frac{1}{3} u^{\frac{3}{2}} + C = -\frac{1}{3} (1 - x^2)^{\frac{3}{2}} + C \]

40. Find the antiderivative of: \( x^3 + \frac{2}{x^2} \).

We need to find the function such that its derivative is the given function. We proceed by differentiating in reverse, that is we add one to the exponent and multiply by the reciprocal of the new exponent. So the antiderivative is
\[ \frac{1}{4} x^4 - \frac{2}{x} + C \]

41. Find the antiderivative of \( 6x(3x^2 + 2)^4 \).

This time notice that the derivative of the function inside the parenthesis is \( 6x \) which is the term out front so we see that we have the chain rule in reverse and we only have to concern ourselves with the function raised to the fourth. So the antiderivative is
\[ \frac{1}{5} (3x^2 + 2)^5 + C \]

and we make the derivative of this function to check we actually found the antiderivative.
42. (This kind of problems won’t be on the test) Evaluate the sum \( \sum_{i=1}^{n}(7i + 2) \).

\[
\sum_{i=1}^{n}(7i + 2) = 7\sum_{i=1}^{n}i + \sum_{i=1}^{n}2 = \frac{7n(n + 1)}{2} + 2n = \frac{7n^2 + 11n}{2}
\]

43. Compute

\[
\int x\sqrt{4-x^2}dx.
\]

Let \( u = 4 - x^2 \), then \(-\frac{1}{2}du = xdx\). So

\[
\int x\sqrt{4-x^2}dx = \frac{1}{2}\int \sqrt{u}du = -\frac{1}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(4 - x^2)^{\frac{3}{2}} + C
\]

44. Evaluate

\[
\int \sqrt{1-3x^2}dx.
\]

For this problem we will use \#42 from the table in the back of the book. We want to get this integral in the form of the square root of something squared minus \( x^2 \). So let us factor out a 3 from under the square root. Now we have

\[
\int \sqrt{1-3x^2}dx = \int \sqrt{\left(\frac{1}{\sqrt{3}} - \frac{x}{\sqrt{3}}\right)^2} - x^2dx = \sqrt{3} \int \left(\frac{1}{\sqrt{3}}\right)^2 - x^2dx
\]

then we can apply \#42 from the table with \( a = \frac{1}{\sqrt{3}} \). So

\[
\sqrt{3} \int \left(\frac{1}{\sqrt{3}}\right)^2 - x^2dx = \sqrt{3}\left(\frac{1}{2} \frac{1}{\sqrt{3}}\right)Sin^{-1}(\sqrt{3}x) + \frac{1}{2} x \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - x^2} + C.
\]

45. Compute

\[
\int_{0}^{1} \sqrt{1-4x^2}dx.
\]

This problem is very similar to the previous one except we have 4 instead of 3, so we proceed in the same way and replace 3 with 4 and evaluate at the limits. So we have

\[
\int_{0}^{\frac{1}{2}} \sqrt{1-4x^2}dx = 2\left(\frac{1}{2}\right)^{\frac{1}{4}} Sin^{-1}(2x) + \frac{1}{2} x \sqrt{\left(\frac{1}{2}\right)^2 - x^2}\bigg|_{0}^{\frac{1}{2}} = simplefy/
\]

46. Compute

\[
\int x^2\sqrt{1+x^3}dx.
\]

Let \( u = 1 + x^3 \) then \( \frac{1}{3}du = x^2dx \). Then

\[
\int x^2\sqrt{1+x^3}dx = \frac{1}{3} \int \sqrt{u}du = \frac{2}{9} u^{\frac{3}{2}} + C = \frac{2}{9}(1 + x^3)^{\frac{3}{2}} + C
\]
47. Compute 
\[ \int \sin x \cos x \, dx. \]

Let \( u = \sin x \), then \( du = \cos x \, dx \) and so 
\[ \int \sin x \cos x \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 x + C \]

48. Compute 
\[ \int \frac{x}{1 - 3x^2} \, dx. \]

Let \( u = 1 - 3x^2 \), so \(-\frac{1}{6} \, du = x \, dx\). Thus 
\[ \int \frac{x}{1 - 3x^2} \, dx = -\frac{1}{6} \int \frac{1}{u} \, du = -\frac{1}{6} \ln |u| + C = -\frac{1}{6} \ln |1 - 3x^2| + C \]

notice when we take the antiderivative of \( \frac{1}{u} \) we get the natural log and we may only 
take the natural log of positive numbers and hence must use the absolute value of the argument.

49. Compute 
\[ \int \frac{x}{5 - x^2} \, dx. \]

Set \( u = 5 - x^2 \) then \( du = -2x \, dx \).

The answer to 49 is then \( -\frac{1}{2} \ln(5 - x^2) + C \)

50. Compute 
\[ \int \frac{1}{3 - x^2} \, dx. \]

Factor the denominator by \( 3 - x^2 = (x - \sqrt{3})(x + \sqrt{3}) \). Using the table we have 
\[ \frac{1}{2\sqrt{3}} \ln \frac{|x - \sqrt{3}|}{|x + \sqrt{3}|}. \]

51. Compute 
\[ \int_0^2 \frac{1}{1 + 9x^2} \, dx. \]

The antiderivative of the function is \( \frac{1}{3} \text{Tan}^{-1}(3x) \) using the fundamental theorem of 
calculus gives us the solution to the definite integral is \( \frac{1}{3} \text{Tan}^{-1}(6) - \frac{1}{3} \text{Tan}^{-1}(0) = \frac{1}{3} \text{Tan}^{-1}(6) \).
52. Compute
\[ \int_0^2 \frac{x}{1 + 4x^2} \, dx. \]

The antiderivative is the function is \( \frac{1}{8} \ln(1 + 4x^2) \) The F.T.o.C. tells us that the solution to the definite integral is then \( \frac{1}{8} \ln(37) - \frac{1}{8} \ln(1) = \frac{1}{8} \ln(37). \)

53. Compute
\[ \int_0^1 xe^{-x^2} \, dx. \]

The antiderivative is the function is \( -\frac{1}{2} e^{-x^2} \) The F.T.o.C. tells us that the solution to the definite integral is then \( -\frac{1}{2} e^{-1} - \frac{1}{2} e^0 = -\frac{1}{2e} + \frac{1}{2}. \)

54. Compute
\[ \int_{-1}^1 xe^{-x^2} \, dx. \]

The antiderivative is the function is \( -\frac{1}{2} e^{-x^2} \) The F.T.o.C. tells us that the solution to the definite integral is then \( -\frac{1}{2} e^{-1} - \frac{1}{2} e^1 = 0 \)

55. Compute
\[ \int_0^1 xe^{-4x^2} \, dx. \]

The antiderivative is the function is \( \frac{1}{8} e^{-4x^2} \) The F.T.o.C. tells us that the solution to the definite integral is then \( \frac{1}{8} e^{-1} - \frac{1}{8} e^0 = -\frac{1}{2e^4} + \frac{1}{2}. \)

56. Compute
\[ \int_0^1 xe^{-\frac{x^2}{5}} \, dx. \]

The antiderivative is the function is \( \left( -\frac{5}{2} \right) e^{-\frac{x^2}{5}} \) The F.T.o.C. tells us that the solution to the definite integral is then \( \left( -\frac{5}{2} \right) e^{-\frac{1}{5}} - \left( -\frac{5}{2} \right) e^0 = -\frac{5}{2e^{\frac{1}{5}}} + \frac{5}{2}. \)

57. Find the area of the region bounded by the curve \( y = \sin x \) and \( x- \) axis with \( 0 \leq x \leq \pi \)

Since the antiderivative of \( \sin x \) is \( -\cos x \), FTC tells us that this is \( -\cos(\pi) + \cos(0) = 2. \)
58. Find the area of the ellipse $x^2 + \frac{y^2}{4} = 1$.

The area of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$. Here $a = 1$ and $b = 2$. So our area is $2\pi$.

59. Find $\sin^{-1}(0.5), \tan^{-1}(-1)$.

$\sin^{-1}(0.5) = \frac{\pi}{6}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$

60. The differential $dy$ is $dy = f'(x) \, dx$. So for $\sin(x^2)$ we have: $dy = \cos(x^2) \cdot 2x \, dx$ and for $xe^x$ we have $dy = (e^x + xe^x) \, dx$. 