Sample Final Exam A

Please write your answers in the exam books provided.

1. X_1, X_2 and X_3 are independent random variables with means μ_1, μ_2, and μ_3 and variances σ_1^2, σ_2^2, and σ_3^2. Let $Y_1 = X_1 + X_3$ and $Y_2 = X_2 + X_3$.

   (a) Find the means and variances of Y_1 and Y_2.

   (b) Find the covariance of Y_1 and Y_2.

   (c) Suppose $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$. Find the correlation of Y_1 and Y_2.

2. The number of customers who shop at a store on a particular day has a Poisson distribution with mean λ. The amount each customer spends at the store is a random variable with mean μ and variance σ^2. Assume that the customers act independently. The total sales of the store for the day is the total amount spent by all the customers who shop at the store on that day.

   (a) Find the conditional mean and variance of the total sales for the day, given that the number of customers who shop at the store on the day is n.

   (b) Find the unconditional mean and variance of the total sales for the day.

3. Let X_1, X_2 be independent Bernoulli variables with success probability p and let $T = X_1 + X_2$.

   (a) Find the conditional expectation $E[X_1|T = t]$ for $t = 0, 1, 2$.

   (b) Suppose g is a function on $\{0, 1, 2\}$ with the property that $E[g(T)] = 0$ for all $p \in [0, 1]$. Show that $g(t) = 0$ for all $t \in \{0, 1, 2\}$. Hint: Write out $E[g(T)]$ in terms of the PMF of T and consider what it means for this expression to equal zero for all values of $p \in [0, 1]$.

4. The variables U and V are uniformly distributed over the set

   $$\mathcal{A} = \{(u, v) : 0 \leq u \leq 1, v^2 \leq -4u^2 \log u\}$$

   The area of this set is $|A| = \sqrt{\pi/2}$. Find the density of $X = V/U$.

5. Let $X_2, X_3, X_4, \ldots$ be independent Bernoulli variables with $P(X_k = 1) = \frac{1}{k}$. Let $Y_1 \equiv 1$, and for $n = 2, 3, 4, \ldots$ define

   $$Y_n = \begin{cases} n & \text{if } X_n = 1 \\ Y_{n-1} & \text{if } X_n = 0 \end{cases}$$

   What is the marginal distribution of Y_n? Hint: See what the distribution is for $n = 2, 3$ and proceed by induction.

Some Distributions

Bernoulli(p)	
pmf	$P(X = x \mid p) = p^x(1-p)^{1-x}$; $x = 0, 1$; $0 \le p \le 1$
mean, variance	$E[X] = p$, $\mathrm{Var}(X) = p(1-p)$
mgf	$M_X(t) = (1-p) + pe^t$
Binomial(n, p)	
pmf	$P(X = x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}$; $x = 0, 1, \ldots, n$; $0 \le p \le 1$
mean, variance	$E[X] = np$, $\mathrm{Var}(X) = np(1-p)$
mgf	$M_X(t) = ((1-p) + pe^t)^n$
Poisson(λ)	
pmf	$P(X = x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$; $x = 0, 1, \ldots$; $0 \le \lambda < \infty$
mean, variance	$E[X] = \lambda$, $\mathrm{Var}(X) = \lambda$
mgf	$M_X(t) = e^{\lambda(e^t - 1)}$
Geometric(p)	
pmf	$P(X = x \mid p) = p(1-p)^{x-1}$; $x = 1, 2, \ldots$; $0 \le p \le 1$
mean, variance	$E[X] = \frac{1}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$
mgf	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$
Beta(α, β)	
pdf	$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$; $0 < x < 1$
mean, variance	$E[X] = \frac{\alpha}{\alpha+\beta}$, $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Cauchy(θ, σ)	
pdf	$f(x \mid \theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}$; $-\infty < x < \infty$; $-\infty < \theta < \infty$; $\sigma > 0$
mean, variance	do not exist
mgf	does not exist
Gamma(α, β)	
pdf	$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$; $0 < x < \infty$; $\alpha, \beta > 0$
mean, variance	$E[X] = \alpha\beta$, $\mathrm{Var}(X) = \alpha\beta^2$
mgf	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha$, $t < \frac{1}{\beta}$
Normal(μ, σ^2)	
pdf	$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}$; $\sigma^2 > 0$
mean, variance	$E[X] = \mu$, $\mathrm{Var}(X) = \sigma^2$
mgf	$M_X(t) = \exp\{\mu t + \frac{1}{2}t^2\sigma^2\}$

Solutions

1. (a) The means and variances are

$$\begin{aligned} E[Y_1] &= \mu_1 + \mu_3 & \quad \mathrm{Var}(Y_1) &= \sigma_1^2 + \sigma_3^2 \\ E[Y_2] &= \mu_2 + \mu_3 & \quad \mathrm{Var}(Y_2) &= \sigma_2^2 + \sigma_3^2 \end{aligned}$$

(b) The covariance is

$$\begin{aligned} \mathrm{Cov}(Y_1, Y_2) &= \mathrm{Cov}(X_1 + X_3, Y_2) = \mathrm{Cov}(X_1, Y_2) + \mathrm{Cov}(X_3, Y_2) \\ &= \mathrm{Cov}(X_1, X_2 + X_3) + \mathrm{Cov}(X_3, X_2 + X_3) \\ &= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_3, X_2) + \mathrm{Cov}(X_3, X_3) \\ &= 0 + 0 + 0 + \mathrm{Var}(X_3) = \sigma_3^2 \end{aligned}$$

(c) If $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$ then $\mathrm{Var}(Y_1) = \mathrm{Var}(Y_2) = 2\sigma^2$ and $\mathrm{Cov}(Y_1, Y_2) = \sigma^2$. So the correlation is

$$\rho = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}} = \frac{\sigma^2}{2\sigma^2} = \frac{1}{2}$$

2. Let N be the number of customers, X_i the amount purchased by customer i, and $T = X_1 + \dots X_N$ the total sales.

(a) Given $N = n$, the total sales $T = X_1 + \cdots + X_n$ is a sum of n independent individual purchase amounts. So $E[T|N = n] = n\mu$ and $\mathrm{Var}(T|N = n) = n\sigma^2$.

(b) The mean and variance of N are both equal to λ. So the expected total sales is $E[T] = E[E[T|N]] = E[N\mu] = E[N]\mu = \lambda\mu$ and the variance of the total sales is

$$\begin{aligned} \mathrm{Var}(T) &= E[\mathrm{Var}(T|N)] + \mathrm{Var}(E[T|N]) = E[N\sigma^2] + \mathrm{Var}(N\mu) \\ &= E[N]\sigma^2 + \mathrm{Var}(N)\mu^2 = \lambda\sigma^2 + \lambda\mu^2 = \lambda(\sigma^2 + \mu^2) \end{aligned}$$

3. (a) If $T = 0$ then $X_1 = X_2 = 0$, so $E[X_1|T = 0] = 0$. Similarly, if $T = 2$ then $X_1 = X_2 = 1$ and $E[X_1|T = 2] = 1$. If $T = 1$ then

$$\begin{aligned} P(X_1 = 1|T = 1) &= \frac{P(X_1 = 1, T = 1)}{P(T = 1)} = \frac{P(X_1 = 1, X_2 = 0)}{P(T = 1)} \\ &= \frac{P(X_1 = 1)P(X_2 = 0)}{P(T = 1)} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2} \end{aligned}$$

and thus $E[X_1|T = 1] = \frac{1}{2}$. So $E[X_1|T] = T/2$.

(b) The expected value $E[g(T)]$ is

$$\begin{aligned} E[g(T)] &= g(0)(1-p)^2 + 2g(1)p(1-p) + g(2)p^2 \\ &= (g(2) + g(0) - 2g(1))p^2 + 2(g(1) - g(0))p + g(0) \end{aligned}$$

This is a quadratic function of p. If this function is zero for all values of $p \in [0,1]$ then all three coefficients must be zero; this implies that $g(0) = g(1) = g(2) = 0$.

4. Let $Y = U$. The range of possible X values is $(-\infty, \infty)$. For a fixed value of $X = x$, the possible values of Y are $[0, e^{-\frac{1}{4}x^2}]$. The inverse transformation is

$$\begin{aligned} U &= Y \\ V &= XY \end{aligned}$$

and the Jacobian determinant of this inverse transformation is

$$\det \begin{pmatrix} 0 & 1 \\ y & x \end{pmatrix} = -y$$

The joint density of X, Y is thus

$$f_{X,Y}(x,y) = \begin{cases} \sqrt{\frac{2}{\pi}}y & 0 < y < e^{-\frac{1}{4}x^2} \\ 0 & \text{otherwise} \end{cases}$$

and the marginal density of X is

$$f_X(x) = \sqrt{\frac{2}{\pi}} \int_0^{e^{-\frac{1}{4}x^2}} y dy = \sqrt{\frac{2}{\pi}} \frac{y^2}{2}\Bigg|_0^{e^{-\frac{1}{4}x^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Thus X has a standard normal distribution. This is the basis of the *ratio of uniforms* method for generating normal variates on a computer.

5. For $n = 2$ we have $P(Y_2 = 2) = \frac{1}{2}$ and $P(Y_2 = 1) = \frac{1}{2}$. For Y_3 we have $P(Y_3 = 3) = P(X_3 = 1) = \frac{1}{3}$ and for $k = 1, 2$

$$P(Y_3 = k) = P(X_3 = 0, Y_2 = k) = P(X_3 = 0)P(Y_2 = k) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

This suggests that Y_n is uniform on its possible values $1, \dots, n$.

Suppose that Y_i is uniformly distributed on $1, \dots, i$ for $i = 1, \dots, n-1$. Then $P(Y_n = n) = P(X_n = 1) = \frac{1}{n}$, and for $k = 1, \dots, n-1$

$$\begin{aligned} P(Y_n = k) &= P(X_n = 0, Y_{n-1} = k) = P(X_n = 0)P(Y_{n-1} = k) \\ &= \frac{n-1}{n} \times \frac{1}{n-1} = \frac{1}{n} \end{aligned}$$

Thus, by induction, Y_n is uniform on $1, \dots, n$ for all $n = 1, 2, \dots$.