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A NOTE ON Q-ORDER OF CONVERGENCE *

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Abstract.

To complement the property of Q-order of convergence we introduce the notions of Q-superorder and Q-suborder of convergence. A new definition of exact Q-order of convergence given in this note generalizes one given by Potra. The definitions of exact Q-superorder and exact Q-suborder of convergence are also introduced. These concepts allow the characterization of any sequence converging with Q-order (at least) 1 by showing the existence of a unique real number $q \in [1, +\infty]$ such that either exact Q-order, exact Q-superorder, or exact Q-suborder q of convergence holds.

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1 Introduction.

We consider sequences $\{x_k\}_{k\geq 0}$ in a metric space (X, d) converging to an element $x^* \in X$, i.e., satisfying

$$\lim_{k\to\infty} d(x_k, x^*) = 0.$$

Certain sequences, most interestingly coming from the application of algorithms in optimization or for the solution of systems of nonlinear equations [1, 2, 3, 4, 5, 6, 7, 8, 10], can be shown to possess the property of converging with Q-order (at least) $q \geq 1$ (the letter Q standing for the word *quotient*). For example, it is well-known that the successive iterates of Newton's method applied to a system of nonlinear equations converge locally (at least) Q-quadratically under some smoothness assumptions.

The purpose of this note is twofold. First, we are motivated to address some discrepancies found in the literature related to the notion of Q-order of convergence. We give here some new, unifying, and more general definitions. The second objective of this note is to give a precise characterization of any sequence converging with Q-order (at least) 1. We show the existence of a unique real number $q \in [1, +\infty]$ such that either exact Q-order, exact Q-superorder, or exact Q-suborder q of convergence holds.

After discussing the notion of Q-order of convergence in Section 2, we give in Section 3 a precise definition of the concepts of Q-superorder and Q-suborder of

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convergence. Then, in Section 4 we introduce the notions of exact Q-order, exact Q-superorder, and exact Q-suborder of convergence. The definition of exact Q-order of convergence given here is shown to be more general than one given by Potra in [8]. Finally, we characterize any sequence converging with Q-order (at least) 1 by showing the existence of a unique real number $q \in [1, +\infty]$ such that either exact Q-order, exact Q-superorder, or exact Q-suborder q of convergence holds.

2 Q-order of convergence.

Let (X, d) be a metric space. For a normed space $(X, \|\cdot\|)$ we consider the standard distance given by $d(x, y) := \|x - y\|$. The notion of Q-order of convergence is concerned with the asymptotic rate of decrease of the distance of a sequence towards its limit.

DEFINITION 2.1. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-order (at least) $q \geq 1$ if there exist two constants $\beta_q \geq 0$ and $K_q \geq 0$ such that for all $k \geq K_q$ we have*

$$d(x_{k+1}, x^*) \leq \beta_q \left(d(x_k, x^*)\right)^q . \tag{2.1}$$

This is equivalent to saying $d(x_{k+1}, x^*) = O\left((d(x_k, x^*))^q\right)$ for $k \to \infty$.

For $q = 2, 3$ the convergence is said to be (at least) Q-quadratic, Q-cubic respectively.

For $q = 1$ the expression *Q-linear convergence* is often reserved in the literature to the situation where $0 \leq \beta_1 < 1$ in (2.1) [1, 5, 6] whereas the situation $\beta_1 \geq 1$ is referred to as *Q-sublinear convergence* [7]. Note that this distinction generally depends directly on the choice of the distance or norm. Consider the sequence in $\mathbf{R}^2$ given by

$$x_{2k} = (2^{-2k}, 0)^T, \qquad x_{2k+1} = (0, 2^{-2k})^T$$

and the norms

$$\|(x_1, x_2)^T\|_A := 2|x_1| + |x_2|, \qquad \|(x_1, x_2)^T\|_B := |x_1| + 2|x_2|.$$

For the A-norm we have

$$\|x_{2k+1}\|_A = 2^{-2k} = 2^{-1}\|x_{2k}\|_A \quad \text{and} \quad \|x_{2k+2}\|_A = 2^{-2k-1} = 2^{-1}\|x_{2k+1}\|_A.$$

Hence, for this sequence we have Q-linear convergence to $x^* = (0, 0)^T$ in the A-norm with $\beta_1 = 1/2$. For the B-norm we have $\|x_{2k+1}\|_B = 2^{-2k+1} = 2\|x_{2k}\|_B$ and $\|x_{2k+2}\|_B = 2^{-2k-2} = 2^{-3}\|x_{2k+1}\|_B$. Hence, in the B-norm we have Q-sublinear convergence with constant $\beta_1 = 2$. Yet, it would make sense to say that this sequence converge Q-linearly independently of the choice of the norm. However, to be consistent with the literature we will not use the expression Q-linear convergence in this situation, but we will instead refer to as Q-order 1 of convergence which is synonymous to either Q-linear or Q-sublinear convergence.

In this article when using the notion of Q-order we will always assume the sequence $\{x_k\}_{k\geq 0}$ to be convergent. Otherwise a divergent sequence like $x_k = k$ for $k \geq 0$ would satisfy the definition of Q-order with $x^* = 0$ for any value of $q \geq 1$. Notice that it is not necessary for a convergent sequence to have a certain Q-order of convergence as the example $x_{2k} := 1/(2k+1)^2$, $x_{2k+1} := 1/(2k+1)$ shows. For this example there is even no finite value of β_1 satisfying (2.1) for $q = 1$.

The notion of Q-order (at least) q of convergence generally depends on the distance d in the metric space (X, d). However, in $\mathbf{R}^n$ by equivalence of norms it is clearly independent of the choice of the norm.

3 Q-superorder and Q-suborder of convergence.

Our first aim in this section is to generalize the properties of Q-superlinear and Q-superquadratic convergence by introducing the notion of Q-superorder of convergence:

DEFINITION 3.1. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-superorder (at least) $q \geq 1$ if for any $\beta_q > 0$ there exists a constant $K_{\beta_q} \geq 0$ such that inequality* (2.1) *holds for all $k \geq K_{\beta_q}$. For $q = 1, 2, 3$ the convergence is said to be (at least) Q-superlinear* [2]*, Q-superquadratic, Q-supercubic respectively.*

We suggest that *Q-order of superconvergence* is synonymous terminology. The above definition is consistent with the literature on Q-superlinear, Q-superquadratic, and Q-supercubic convergence, which we suggest, are synonymous with *Q-linear, Q-quadratic, and Q-cubic superconvergence* respectively. An equivalent characterization of Q-superorder of convergence is given as follows:

LEMMA 3.1. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-superorder (at least) $q \geq 1$ if and only if there exist a sequence of real numbers $\{\beta_k\}_{k\geq 0}$ converging to* 0 *and a constant $K \geq 0$ such that for all $k \geq K$ we have*

$$d(x_{k+1}, x^*) \leq \beta_k \left(d(x_k, x^*)\right)^q .$$

The proof is trivial.

We also have:

LEMMA 3.2. *A sequence $\{x_k\}_{k\geq 0}$ with $x_k \neq x^*$ for all $k \geq k_0$ converges to x^* with Q-superorder (at least) $q \geq 1$ if and only if*

$$d(x_{k+1}, x^*) = o\left((d(x_k, x^*))^q\right) \quad \text{for } k \to \infty.$$

Another similar equivalent characterization is given in Corollary 4.3 below. From Definition 3.1 we can easily show:

LEMMA 3.3. *If a sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-superorder (at least) $q \geq 1$ then it converges to x^* with Q-order (at least) $q \geq 1$.*

The proof is straightforward. The reverse is obviously not true in general. As a counterexample we can consider the real sequence given recursively by $x_{k+1} := x_k^2$ starting from $x_0 := 1/2$. This sequence converges Q-quadratically to $x^* = 0$, but not Q-superquadratically. Nevertheless, we have the following result:

LEMMA 3.4. *If a sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-order (at least) $q > 1$ then it also converges to x^* with Q-superorder (at least) $\bar{q}$ for any $\bar{q}$ satisfying $1 \leq \bar{q} < q$.*

PROOF. Simply consider $(d(x_k, x^*))^q = (d(x_k, x^*))^{(q-\bar{q})+\bar{q}}$. □

It is obviously not necessary for a Q-superlinear sequence to satisfy the condition of Lemma 3.4 for a certain $q > 1$. For example the real sequence defined by $x_k := 1/k!$ which satisfies $x_{k+1} := x_k/(k+1)$ is Q-superlinearly convergent to $x^* = 0$ in the sense of Definition 3.1, but there is no value $q > 1$ satisfying the condition of Lemma 3.4.

A sequence $\{x_k\}_{k\geq 0}$ converging to x^* satisfying for example $d(x_{2k}, x^*) = \exp(-2^{2k}) \cdot 2k$ and $d(x_{2k+1}, x^*) = \exp(-2^{2k+1})/(2k+1)$, has the property of converging with Q-order (at least) $\bar{q}$ and even Q-superorder (at least) $\bar{q}$ for any $\bar{q} < 2$, but not for $\bar{q} = 2$. We will refer to such a sequence as being exactly Q-subquadratically convergent (see Section 4). First we define the property of Q-suborder of convergence as follows:

DEFINITION 3.2. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-suborder (at least) $q > 1$ if it converges with Q-order (at least) $\bar{q}$ for any $\bar{q}$ satisfying $1 \leq \bar{q} < q$. For $q = 2, 3$ the convergence is said to be (at least) Q-subquadratic, Q-subcubic respectively.*

This definition is also motivated by Lemma 3.4. We do not include in it the case $q = 1$ which would not seem to be of much interest and which should not be confused with Q-sublinear convergence (see Section 2). From Definition 3.2 and analogously to Lemma 3.3 we have the following trivial result:

LEMMA 3.5. *If a sequence $\{x_k\}_{k\geq 0}$ converges to x^* with Q-order (at least) $q > 1$ then it converges to x^* with Q-suborder (at least) $q > 1$.*

Generally the notions of Q-superorder and Q-suborder (at least) q of convergence depend on the distance d in the metric space (X, d). However, in $\mathbf{R}^n$ by equivalence of norms they are clearly independent of the choice of the norm.

4 Exact Q-order, exact Q-superorder, and exact Q-suborder of convergence.

We first give a precise definition of exact Q-order of convergence.

DEFINITION 4.1. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with exact Q-order $q \geq 1$ if it converges to x^* with Q-order (at least) q and if it does not converge to x^* with Q-superorder (at least) q.*

The following result states that our definition of exact Q-order q of convergence includes the one given by Potra in [8]:

LEMMA 4.1. *Consider a sequence $\{x_k\}_{k\geq 0}$ converging to x^* with exact Q-order q in the sense of Potra, i.e., such that there exist two positive constants*

$a > 0$ and $b > 0$ satisfying the property

$$a\,(d(x_k, x^*))^q \le d(x_{k+1}, x^*) \le b\,(d(x_k, x^*))^q\,.$$

Then the sequence has exact Q-order q of convergence in the sense of Definition 4.1.

The proof is trivial and left to the reader. Note that the reverse is not true in general. There are sequences converging with exact Q-order q in the sense of Definition 4.1, but not in the sense of Potra. Consider for example in $\mathbf{R}$ the sequence given recursively by

$$x_{2k+1} := \frac{x_{2k}^q}{2k+1},\ x_{2k+2} := x_{2k+1}^q \text{ for } q \ge 1, \text{ starting from } x_0 = 1. \tag{4.1}$$

This sequence which converges to $x^* = 0$ has exact Q-order of convergence q in the sense of Definition 4.1 (see Lemma 4.2), but not in the sense of Potra.

A way of showing exact Q-order q of convergence is given as follows:

LEMMA 4.2. *Given a sequence $\{x_k\}_{k\ge 0}$, assume that $x_k \neq x^*$ for all $k \ge k_0$. Then we can define for $k \ge k_0$ and $q \ge 1$ the sequence of real numbers*

$$\gamma_{q,k} := \frac{d(x_{k+1}, x^*)}{(d(x_k, x^*))^q}. \tag{4.2}$$

If the sequence $\{\gamma_{q,k}\}_{k\ge k_0}$ remains asymptotically bounded (i.e., there exist two constants $C_q > 0$ and $K_q \ge k_0$ such that $\gamma_{q,k} \le C_q$ for all $k \ge K_q$) then $\{x_k\}_{k\ge 0}$ converges to x^* with Q-order (at least) q. Moreover, if in addition we do not have $\lim_{k\to\infty} \gamma_{q,k} = 0$ then we have exact Q-order q of convergence.

The proof is again trivial and left to the reader. When the limit for $k \to \infty$ in (4.2) exists we have from Lemma 3.1 and Lemma 4.2 the following result:

COROLLARY 4.3. *Given a sequence $\{x_k\}_{k\ge 0}$ with $x_k \neq x^*$ for all $k \ge k_0$, assume that the following limit exists for $q \ge 1$*

$$\gamma_q := \lim_{k\to\infty} \frac{d(x_{k+1}, x^*)}{(d(x_k, x^*))^q}. \tag{4.3}$$

The value γ_q is called the asymptotic constant factor. If $\gamma_q > 0$ we have exact Q-order q of convergence. If $\gamma_q = 0$ then the sequence $\{x_k\}_{k\ge 0}$ converges to x^* with Q-superorder (at least) q.

Note that the asymptotic constant factor γ_q depends on the choice of the distance d. When $\gamma_1 = 0$ for $q = 1$ or $\gamma_2 = 0$ for $q = 2$ in (4.3), this is often taken as a definition of Q-superlinear and Q-superquadratic convergence (see, e.g., [3, 5, 6]) whereas Definition 3.1 is slightly more general since it also encompasses the case where $x_k = x^*$ for $k \ge k_0$. As stated in Lemma 4.2, it is not necessary to have existence of the limit $\gamma_q > 0$ in (4.3) to have exact Q-order q of convergence as the example given in (4.1) shows.

We introduce next the definitions of exact Q-superorder and exact Q-suborder of convergence:

Definition 4.2. *A sequence* $\{x_k\}_{k\geq 0}$ *converges to* x^* *with exact Q-superorder* $q \geq 1$ *if it converges to* x^* *with Q-superorder (at least)* $q \geq 1$ *and it does not converge to* x^* *with Q-suborder (at least)* $\bar{q}$ *for any* $\bar{q} > q$. *If a sequence converges with Q-superorder (at least)* q *for all* $q \geq 1$ *then we say that it converges with exact Q-superorder* $+\infty$.

Definition 4.3. *A sequence* $\{x_k\}_{k\geq 0}$ *converges to* x^* *with exact Q-suborder* $q > 1$ *if it converges with Q-suborder (at least)* q *and it does not converge with Q-order (at least)* q.

Based on the following trivial result there is no need to define an exact Q-order or an exact Q-suborder of convergence being equal to $+\infty$:

Lemma 4.4. *If a sequence converges with Q-order or Q-suborder* q *for all* $q > 1$ *then it converges with exact Q-superorder* $+\infty$.

We state now the main result of this note:

Theorem 4.5. *A sequence* $\{x_k\}_{k\geq 0}$ *converges to* x^* *with Q-order (at least)* 1 *if and only if there exists a unique* $q \in [1, +\infty]$ *such that the sequence converges to* x^* *with either exact Q-order, exact Q-superorder, or exact Q-suborder* q *of convergence (excluding exact Q-suborder* 1 *which is left undefined).*

Proof. Clearly, a sequence converging with exact Q-order, exact Q-superorder, or exact Q-suborder q with $q \in [1, +\infty]$ (excluding exact Q-suborder 1) must converge with Q-order (at least) 1. Showing the reverse implication is the interesting part of this theorem. From Definitions 4.1, 4.2, and 4.3 the notions of exact Q-order, exact Q-superorder, and exact Q-suborder q of convergence are mutually exclusive. We define

$$q := \sup\{\, r \geq 1 \mid \{x_k\}_{k\geq 0} \text{ is of Q-order (at least) } r\} \tag{4.4}$$

which corresponds to the definition of *Q-order* given by Potra in [8]. Note that we could have defined equivalently

$$q := \sup\{\, r \geq 1 \mid \{x_k\}_{k\geq 0} \text{ is of Q-superorder (at least) } r\}$$

or

$$q := \sup\{\, r \geq 1 \mid \{x_k\}_{k\geq 0} \text{ is of Q-suborder (at least) } r\}\,,$$

all these quantities being equal. If $q = +\infty$, the sequence converges with exact Q-superorder $+\infty$. Assume now that q is finite. For this value of q we consider the nonnegative real sequence $\{\gamma_{q,k}\}_{k\geq 0}$ satisfying

$$d(x_{k+1}, x^*) \leq \gamma_{q,k}(d(x_k, x^*))^q$$

and such that for each $k \geq 0$ there is no smaller nonnegative value than $\gamma_{q,k}$ satisfying this inequality. If the sequence $\{\gamma_{q,k}\}_{k\geq 0}$ converges to zero then we have exact Q-superorder q of convergence. If the sequence $\{\gamma_{q,k}\}_{k\geq 0}$ is unbounded then we have exact Q-suborder q of convergence. Otherwise the sequence $\{\gamma_{q,k}\}_{k\geq 0}$ must remain bounded and does not converge to zero, therefore we have exact Q-order q of convergence in this situation. $\square$

We have shown in Theorem 4.5 that only the sequences which do not converge with Q-order (at least) 1 do not possess an exact Q-order, an exact Q-superorder, or an exact Q-suborder of convergence. For sequences converging with Q-order (at least) 1, the value q in (4.4) is also equal to the definition of the Q-order given by Luenberger in [5]:

$$q = \sup \left\{ r \geq 1 \;\middle|\; \{x_k\}_{k\geq 0} \text{ satisfies } \limsup_{k\to\infty} \frac{d(x_{k+1},x^*)}{(d(x_k,x^*))^r} < +\infty \right\}$$

and also to the one given by Ortega and Rheinboldt in [7]

$$q = \inf \left\{ r \geq 1 \;\middle|\; \{x_k\}_{k\geq 0} \text{ satisfies } \limsup_{k\to\infty} \frac{d(x_{k+1},x^*)}{(d(x_k,x^*))^r} = +\infty \right\}.$$

An often more explicit way to compute the value q in (4.4) is by the formula

$$q = \lim_{k\to\infty} \frac{\log(e_{k+1}/e_k)}{\log(e_k/e_{k-1})}$$

for $e_k := d(x_k,x^*)$ whenever the limit on the right-hand side exists, for example when we have the asymptotic behavior $e_{k+1} \approx c e_k^q$ for $k \to \infty$.

Note that the notion of Q-order of convergence is not an absolute measure of the speed of convergence of a sequence, it can also be misleading. For example, consider a first sequence $x_k := 2^{-k}$ converging exactly Q-linearly to $x^* = 0$ and a second sequence given by $y_{2k} := 2^{-(2^{2k})}$ and $y_{2k+1} := 2^{-(3^{2k+1})}$. The sequence $\{y_k\}_{k\geq 0}$ converges more rapidly toward $x^* = 0$ than the sequence $\{x_k\}_{k\geq 0}$ since $|y_k| < |x_k|$ for any $k \geq 0$. However, the sequence $\{y_k\}_{k\geq 0}$ does not even converge with Q-order 1. A slightly weaker form of order of convergence is given by the concept of R-order of convergence (the letter R standing for the word *root*):

Definition 4.4. *A sequence $\{x_k\}_{k\geq 0}$ converges to x^* with R-order (at least) $r \geq 1$ if there exists a sequence of real numbers $\{\beta_k\}_{k\geq 0}$ converging to zero with Q-order (at least) r such that*

$$d(x_k,x^*) \leq \beta_k.$$

The two aforementioned sequences $\{x_k\}_{k\geq 0}$ and $\{y_k\}_{k\geq 0}$ converge to $x^* = 0$ with R-order 1 and 2 respectively. A drawback of the notion of R-order of convergence is that the errors of a sequence resulting from the application of an algorithm characterized by a certain R-order are not necessarily monotonically decreasing. Following Potra and Pták in [9], a more refined concept of order (or rate) of convergence is given by a certain real function instead of a real number, but this is beyond the scope of this note.

5 Conclusion.

In this note we have given new definitions of Q-superorder and Q-suborder of convergence. The definition of Q-superorder of convergence embraces the properties of Q-superlinear and Q-superquadratic convergence. The definition of exact Q-order of convergence given here refines one given by Potra in [8]. The new notions of exact Q-superorder and exact Q-suborder of convergence are also introduced. Theorem 4.5 is the main result of this note and is new. It shows that there exists a unique real number $q \in [1, +\infty]$ characterizing either the exact Q-order, exact Q-superorder, or exact Q-suborder, of any sequence converging with Q-order (at least) 1, in particular of any (at least) Q-linear convergent sequence.

To conclude shortly, the different notions of order of convergence are a means to characterize the asymptotic behavior of the quantities $d(x_k, x^*)$ when $k \to \infty$. Clearly, this information cannot really be completely compressed into one unique number. There is certainly no absolute measure of the speed of convergence of a sequence. Nevertheless, the notions of Q-order, of Q-superorder, and of Q-suborder of convergence already give much insight when analyzing certain numerical algorithms and they are generally satisfactory for this purpose.

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REFERENCES

1. J. E. Dennis and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, SIAM Classics in Appl. Math., Reprinted by SIAM, Philadelphia, PA, 1993.
2. R. Fletcher, *Practical Methods of Optimization*, 2nd ed., Wiley, Chichester, 1987.
3. P. E. Gill, W. Murray, and M. H. Wright, *Practical Optimization*, Academic Press, New York, 1981.
4. C. T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, Vol. 16 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1995.
5. D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd ed., Addison-Wesley, Reading, MA, 1984.
6. J. Nocedal and S. Wright, *Numerical Optimization*, Springer Series in Operations Research, Springer-Verlag, New York, 1999.
7. J. Ortega and W. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.
8. F. A. Potra, *On Q-order and R-order of convergence*, J. Optim. Theory Appl., 63 (1989), pp. 415–431.
9. F. A. Potra and V. Pták, *Nondiscrete Induction and Iterative Processes*, Vol. 103 of Research Notes in Mathematics, Pitman, Boston, MA, 1984.
10. W. Rheinboldt, *Methods for Solving Systems of Nonlinear Equations*, 2nd ed., SIAM Classics in Appl. Math., SIAM, Philadelphia, PA, 1998.