Recent advances in the numerical solution of Fractional Differential Equations

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Abstract

In recent years, the efficient numerical solution of Hamiltonian problems has been tackled by means of *Hamiltonian Boundary Value Methods (HBVMs)* (see, e.g., the monograph [1]). Such methods can be defined through a truncated expansion of the vector field defining the problem,

$$\dot{y}(ch) = f(y(ch)), \qquad c \in [0,1], \qquad y(0) = y_0 \in \mathbb{R}^m,$$

along the Legendre polynomial basis orthonormal on $[0,1]$, $\int_0^1 P_i(c)P_j(c)\mathrm{d}c = \delta_{ij}$, $i,j = 0,1,\ldots$. For a given $s \in \mathbb{N}$, this results into a polynomial vector field $\dot{\sigma}(ch)$, of degree $s-1$, defined as:

$$\dot{\sigma}(ch) = \textstyle\sum_{j=0}^{s-1} P_j(c)\gamma_j(\sigma), \quad c \in [0,1], \quad \sigma(0) = y_0 \in \mathbb{R}^m,$$

$$\gamma_j(\sigma) = \textstyle\int_0^1 P_j(\tau) f(\sigma(\tau h))\mathrm{d}\tau.$$

As is clear [1,2], this is equivalent to require the residual be orthogonal to all polynomials of degree $s-1$, w.r.t. the $L_2[0,1]$ product:

$$\int_0^1 P_j(\tau)\left[\dot{\sigma}(\tau h) - f(\sigma(\tau h))\right]\mathrm{d}\tau = 0, \qquad j = 0,\ldots,s-1.$$

Discretizing the Fourier coefficients $\gamma_j(\sigma)$ by means of a Gauss-Legendre quadrature of order $2k$, $k \geq s$, results into a HBVM(k,s) method. Though formally a k-stage Runge-Kutta method [1], the corresponding discrete problem can be always recast in terms of the s Fourier coefficients, independently of k, so that the accurate approximation of the inetgrals is not an issue. Moreover, the availability of a very efficient nonlinear *blended iteration* for solving the generated discrete problems [1,3], allows using large values of s coupled with huge timesteps h, resulting into *spectrally accurate in time* methods [4]. These latter turn out to be particularly effective when solving large-size semi-linear ODE problems deriving from the space semi-discretization of Hamiltonian PDEs [5].

This approach has been recently extended to cope with the efficient numerical solution of *Fractional Differential Equations*, which we take in the form, for $\alpha \in (0,1)$,

$$y^{(\alpha)}(ch) = f(y(ch)), \qquad c \in [0,1], \qquad y(0) = y_0 \in \mathbb{R}^m,$$

with $y^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} y'(x) \mathrm{d}x$ the Caputo fractional derivative. In such a case, a polynomial vector field, defining a corresponding *Fractional HBVM (FHBVM)*, is derived by considering a truncated expansion of the vector field along an orthonormal Jacobi basis naturally induced by the problem, $\int_0^1 \omega(c) P_i(c) P_j(c) \mathrm{d}c = \delta_{ij}$, $i, j = 0, 1, \ldots$, $\omega(c) = \alpha(1-c)^{\alpha-1}$:

$$\sigma^{(\alpha)}(ch) = \textstyle\sum_{j=0}^{s-1} P_j(c) \gamma_j(\sigma), \quad c \in [0,1], \quad \sigma(0) = y_0 \in \mathbb{R}^m,$$
$$\gamma_j(\sigma) = \textstyle\int_0^1 \omega(\tau) P_j(\tau) f(\sigma(\tau h)) \mathrm{d}\tau.$$

Again, this approximated polynomial vector field is equivalent to requiring that the residual be orthogonal to all polynomials of degree $s-1$, w.r.t. the corresponding *weighted* $L_2[0,1]$ product:

$$\int_0^1 \omega(\tau) P_j(\tau) \left[\sigma^{(\alpha)}(\tau h) - f(\sigma(\tau h)) \right] \mathrm{d}\tau = 0, \qquad j = 0, \ldots, s-1.$$

Discretization of the Fourier coefficients $\gamma_j(\sigma)$ by using a Gauss-Jacobi quadrature of order $2k$, $k \geq s$, then results into a FHBVM(k,s) method (as is clear, this reduces to a HBVM(k,s) method, when $\alpha = 1$) [6]. Also in this case, the availability of an efficient nonlinear iteration, allowing using large values of s, coupled when needed with a graded mesh to cope with possible nonsmooth vector fields, allows reaching spectrally accurate in time solutions. FHBVMs have been implemented into two Matlab codes, `fhbvm` [7] and `fhbvm2` [8]. A systematic comparison on the set of test problems collected in the FDE-Testset [9] clearly shows the effectiveness of such codes, w.r.t. existing Matlab codes available on the web.

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Stroboscopic averaging methods to study systems with autoresonance

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Abstract

Autoresonance is a phenomenon of physical interest that takes place when a nonlinear oscillator automatically adjusts its instantaneous frequency so as to match the varying frequency of a forcing term, causing that the amplitude of the oscillations keeps growing. It may happen even if the frequency of the forcing term varies slowly. The stroboscopic averaging method (SAM), which provides an efficient numerical technique for the integration of highly oscillatory systems, cannot be used directly to study autoresonance due to the slow changes of the forcing frequency. We study how to modify SAM to cater for such slow variations and include numerical experiments for the Duffing oscillator showing the computational advantages of using SAM.

To be announced

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Feedback Control and Collective Dynamics of Microswimmer Swarms

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Abstract

We investigate the control and coordination of artificial microswimmers inspired by breaststroke-like propulsion within a geometric framework. At low Reynolds numbers, locomotion arises from non-reciprocal shape changes, leading naturally to a formulation in terms of sub-Riemannian geometry.
We first analyze the configuration dynamics of a single swimmer. Starting from a three-arm model and reducing to a two-arm subsystem, we show that the resulting control system induces a rank-two distribution with Lie algebra growth corresponding to a Cartan $(2, 3, 5)$ structure. This characterization yields controllability, highlights the role of abnormal extremals, and connects the model to classical articulated swimmers such as the Purcell three-link swimmer. Numerical simulations evaluate propulsion efficiency under different criteria.
Motivated by the need for collective effects at microscopic scales, we study decentralized coordination strategies for swarms of microswimmers. Each agent adjusts its shape variables using only local information exchanged with neighboring agents. The interaction structure is modeled by a fixed communication graph, and the resulting feedback law is formulated using the associated graph Laplacian. We provide convergence estimates for the synchronization dynamics and illustrate the behavior of the swarm through numerical simulations under various interaction scenarios.

On a class of linearly implicit Runge–Kutta methods for reaction–diffusion equations

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Abstract

In recent years, the development and analysis of reaction-diffusion models have gained increasing attention due to their applicability in various sustainability-related contexts. Notable examples include the modeling of ecological dynamics, such as vegetation patterns in semi-arid ecosystems [1]. A further relevant application, in the framework of sustainable innovation, arises in the mathematical modeling of supply chains and production networks, where continuum descriptions based on Partial Differential Equations (PDEs) can be used to represent the evolution of material and product flows across interconnected processors, buffers and distribution nodes [6]. These models are generally described by systems of PDEs that, after space discretization, typically lead to high-dimensional stiff systems of ordinary differential equations, whose efficient and reliable time integration remains a challenging task.

In this talk, we discuss recent developments in Runge–Kutta (RK) methods designed for the numerical solution of such stiff problems. Starting from the framework of Time-Accurate and highly Stable Explicit RK (TASE–RK) methods [3, 4], we present adapted strategies for reaction–diffusion systems. In particular, suitable operators are used to improve the stability properties of explicit schemes without compromising computational efficiency. The resulting General TASE–RK methods are designed to reduce error constants and computational effort for the class of problems under consideration, while getting suitable stability properties such as strong $A(\theta)$-stability [5].

Furthermore, we focus on the derivation of a more flexible class of linearly implicit schemes, referred to as Modified TASE–RK methods [2]. In this setting, different operators may be employed at the internal stages, allowing to move beyond the standard TASE–RK formulation. By exploiting the relation with W-methods, schemes of order up to four are derived with a reduced number of linear systems with respect to classical TASE–RK methods. The proposed strategy also permits the design of strongly A-stable methods, which are suitable for advection–reaction–diffusion models.

The theoretical properties of the presented methods are supported by several numerical experiments.

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Attractivity of the disease-free subspace in epidemic models

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Abstract

We extend the classical stability analysis of disease-free equilibria in epidemiological compartmental problems. In particular, we construct a method to examine whether disease-free orbits attract nearby orbits (disease-free or not).
In the theory of compartmental epidemic dynamical models, a clear distinction is made between infected compartments and disease-free compartments. Typically, the behavior of the system is studied in the neighborhood of a disease-free equilibrium. Instead, we subdivide the space of disease-free states into regions where they attract nearby orbits and regions where they do not attract them. More precisely, we subdivide the space of disease-free states into regions where either (a) all eigenvalues of the Jacobian of the system have negative real parts, or (b) at least one eigenvalue has a positive real part, or (c), the border case: not in (a) or (b). In general, this is a difficult task. But extensive numerical tests and graphical illustrations show the practical applicability of our method in understanding the topology of the phase portraits and their global behavior.
We consider some classically important models and hypothesize that the same results hold for a large class of epidemic models.

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Efficient Simulations of a Pitting Corrosion Model

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Abstract

In recent years, the development of efficient numerical methods for large-scale reaction-diffusion systems has become increasingly important in several sustainability-oriented industrial applications. In particular, predictive models for material degradation and corrosion processes play a key role in improving the reliability and lifetime of engineering infrastructures, supporting maintenance strategies, and reducing disruptions in production and supply-chain systems. The mathematical models arising in these contexts are often governed by partial differential equations [4].

In this talk, we focus on the simulation of corrosion processes through a phase-field model for metal pitting corrosion [5]. Phase-field approaches are especially attractive because they naturally describe the evolution of corrosion fronts and complex moving interfaces without requiring costly remeshing techniques. Nevertheless, the resulting equations are highly stiff, making standard explicit time integrators inefficient due to severe stability constraints [2, 3].

To overcome these issues, we propose a matrix-oriented strategy that exploits the Kronecker sum structure of the discrete Laplacian operator arising from finite difference discretizations on rectangular domains. This allows the construction of efficient schemes that avoid large matrix inversions while remaining applicable also to non-rectangular geometries [1].

The proposed methods are assessed through numerical experiments in two- and three-dimensional settings and compared with existing approaches. The results show significant improvements in computational efficiency together with accurate resolution of the corrosion dynamics, highlighting the potential of the proposed methodology for fast and reliable corrosion simulations in realistic engineering applications [1].

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To be announced

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To be announced

References

Adaptive FEM with Explicit Time Integration For the Wave Equation

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Abstract

Starting from a recent a posteriori error estimator for the finite element solution of the wave equation with explicit time integration [Grote, Lakkis, Santos, 2024], we devise a space-time adaptive strategy which includes both time evolving meshes and local time-stepping to overcome any overly stringent CFL stability restriction on the time-step due to local mesh refinement. Moreover, at each time-step the adaptive algorithm monitors the accuracy thanks to the error indicators and recomputes the current step on a refined mesh until the desired tolerance is met; meanwhile, the mesh is coarsened in regions of smaller errors. Leapfrog based local time-stepping is applied in all regions of local mesh refinement to incorporate adaptivity into fully explicit time integration with mesh change while retaining efficiency. Numerical results illustrate the optimal rate of convergence of the a posteriori error estimators on time evolving meshes.

Solving time fractional differential equations with standard stiff integrators

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Abstract

A fractional differential equation is a generalization of an integer order differential equation. The (local) integer derivative operator is replaced by a (non-local) fractional derivative. With the help of the Riemann-Liouville fractional integral such problems are transformed into Volterra integro-differential equations.
The main idea of our numerical approach is to approximate the fractional kernel by a sum of exponential functions, and then to transform the fractional integral (of convolution type) into a set of ordinary differential equations. Solving the resulting (stiff) differential equation by a numerical time integrator gives a fast and memoryless numerical method.
It is explained how the code RADAU5 can be used for solving fractional differential equations. Numerical experiments illustrate the accuracy and the efficiency of the proposed method.

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Efficient Low-Memory Exponentially Fitted Explicit Runge–Kutta Schemes

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Abstract

This talk addresses the construction and analysis of explicit low-storage exponentially fitted Runge–Kutta methods for the efficient numerical solution of high-dimensional oscillatory differential systems.
We investigate schemes that require only two memory registers per dependent variable, focusing on their consistency, convergence, and absolute stability characteristics. Particular attention is paid to the comparison between the Williamson [3] and van der Houwen [4] low-storage families, which involve $(2s-1)$ free parameters, and the more flexible formulations of Ketcheson [5] and Calvo [1], characterized by $(3s-3)$ degrees of freedom.
Based on these frameworks, new optimal third-order three-stage, fourth-order four-stage and fourth-order five-stage methods are derived by combining exponential fitting requirements with accuracy and stability constraints [2].
The performance of the resulting integrators is assessed through a series of numerical experiments, which confirm their effectiveness and computational efficiency when applied to oscillatory problems.

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Various results on the numerical solution of ODEs and DAEs by implicit Runge-Kutta type methods

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Abstract

We discuss various recent results for the time integration of ordinary differential equations (ODEs) and differential-algebraic equations (DAEs) by implicit Runge-Kutta (IRK) type methods.

- First we consider semi-explicit DAEs of index 2. We present some modifications to standard IRK methods to obtain superconvergence of non-stiffly accurate IRK methods, such as Gauss, Radau IA, Radau I, Radau II, Lobatto IIIB, and Lobatto IIIC* methods. Those methods are named *specialized Runge-Kutta methods for index* 2 *DAEs (SRK-DAE2).*
- Second we consider semi-explicit DAEs of index 3 and stiffly accurate IRK methods such as Radau IIA and Lobatto IIIC methods. We present a simple cheap projection trick to preserve some underlying hidden constraints and to reestablish superconvergence of certain differential components (the velocities and momenta in the context of mechanical systems with holonomic constraints). This modification also leads to a clear definition of the corresponding reverse methods Radau I and Lobatto IIIC* methods.
- Third we give some order conditions of starting algorithms for IRK methods for ODEs based on corresponding reverse methods. We also consider starting approximations for SRK-DAE2 methods.
- Fourth we discuss a new approach to time integration based on implicit methods, time-rescaling, a fixed step size, and high order predictors. The use of a fixed step size has several advantages over standard variable step sizes:
  - backward error results can be formulated;
  - global error estimation and extrapolation are possible;
  - no recomputation of Jacobian due to step size change;
  - no hyperparameters for step size selection are needed.

  The possible inefficiency of using a fixed step size h is addressed by the use of time-rescaling functions $\rho(t, x)$ where $h\rho(t, x)$ acts like a variable step size in the original time t.

- Finally if time permits we consider ODEs with stiffness. We present some results about sufficient conditions ensuring local contractivity, hence convergence, of modified Newton iterations of IRK methods with step size conditions independent of stiffness. A seemingly minor but crucial condition on the Jacobian was missed in the literature.

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Time integration of tree tensor networks

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Abstract

The talk first presents some numerical experiments with time-dependent tree tensor network algorithms for the approximation of quantum spin system dynamics. It continues with the basics in the design of time integration methods that are robust to the typical presence of small singular values, that have good structure-preserving properties (norm, energy conservation or dissipation), and that allow for rank adaptivity and for parallelism. This discussion of basic concepts forms the main part of the talk and will be done for the smallest possible type of tensor network differential equations, namely low-rank matrix differential equations, which are of interest in their own right. Once this nontrivial technically simplest case is understood, there is a systematic path to the extension of the integrators and their favourable properties to general tree tensor networks.

This talk is based on joint work with many colleagues and former and present students, among which I wish to single out Othmar Koch for the first mathematical work on dynamical low-rank approximation (DLRA) in 2007, Ivan Oseledets for jointly discovering the first robust DLRA integrator in 2014 (the projector-splitting integrator), Gianluca Ceruti and Jonas Kusch for jointly developing the Basis Update & Galerkin (BUG) integrators since 2022, and Hanna Walach, Gianluca Ceruti, Dominik Sulz and Charlotte Verhoeven for the recent systematic extension from low-rank matrices to general tree tensor networks both in theory and in increasingly efficient implementations.

Stability of Numerical Methods Depending on a Matrix Parameter

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Abstract

The stability analysis of numerical time integration methods is often carried out under the assumption that the matrix parameter involved in the scheme coincides with the exact Jacobian of the continuous problem. However, in many practical situations this requirement makes the numerical method very expensive or it is too restrictive. The matrix used by the numerical method may be frozen, approximated, simplified, preconditioned, or chosen in order to reduce the computational cost, and therefore it does not necessarily represent the exact Jacobian of the vector field.
In this talk we discuss stability issues for numerical methods whose formulation depends on a matrix parameter A, which may differ from the Jacobian J. The central idea is to understand how the mismatch between A and J affects the linear stability properties of the method. This point of view is motivated by recent results on TASE–Runge–Kutta methods with inexact Jacobian, where the decomposition $J = A + B$ allows one to derive stability criteria that take into account both the stabilizing contribution of A and the residual part B. We also discuss related questions for multistep exponential time difference methods, where the matrix parameter plays an analogous role in the definition of the numerical propagator.
The goal is to provide stability conditions that are not tied to the unrealistic assumption $A = J$, but instead remain meaningful for implementable schemes in which A is selected according to efficiency, robustness, or structural considerations.

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Explicit Runge-Kutta methods with reused stages

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Abstract

Explicit Runge–Kutta (RK) methods for the numerical solution of IVPs in differential equations can reduce the computational cost if the first stage of one step is substituting by the last stage of the previous step. Thus, except for the first step, the computational cost of the s-stage RK is reduced from s to $(s-1)$ function evaluations per step. In general, if a RK method has order p, the method reusing the last stage has order smaller than p. We will study the conditions that the coefficients of an explicit RK method with order p must satisfy so that the error of the method reusing the last stage has the same order p of accuracy. In particular, we will see that there exist methods with s stages and order s for which the methods reusing the last stage have also order s. Numerical experiments will be presented to confirm the theoretical results.

Conditional stability of the geodesic Euler method on Riemannian manifolds with applications to neural network architectures

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Abstract

Numerical integrators for differential equations on manifolds are well established, but their stability and convergence properties are less developed. In a recent paper we found accurate bounds on the time step for the geodesic Euler method to be non-expansive in the case of constant curvature spaces. We will briefly review these results and also discuss some work in progress on neural network architectures which admit non-expansive layer maps on Hadamard manifolds.

New W-methods for multidimensional advection–diffusion–reaction problems

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Abstract

In this talk, we focus on the time integration of semi-discretized multidimensional Partial Differential Equations (PDEs). In particular, the aim is to derive efficient numerical schemes that exploit the algebraic structure of the discrete operators.
First, we construct new linearly implicit time integrators for stiff systems arising from the spatial semi-discretization of PDEs [4]. The proposed schemes are developed within the framework of W-methods and are inspired by the so-called Time-Accurate and highly Stable Explicit Runge–Kutta (TASE–RK) methods [3]. We theoretically analyze the consistency and stability of the new methods, referred to as TASE W-methods, showing that they are characterized by low error constants, good stability properties, and a reduced computational cost with respect to standard linearly implicit time integrators.
Particular attention is then devoted to the adaptation of the new W-methods to multidimensional PDEs, through suitable strategies that take advantage of the structure of the spatial discretization. On the one hand, techniques based on Approximate Matrix Factorization (AMF) are considered to exploit the directional splitting naturally induced by multidimensional differential operators [1]. On the other hand, when the discretized problem exhibits a tensor-product structure, matrix-oriented implementations are investigated as an alternative strategy to reduce memory requirements and computational cost [2]. These two approaches provide complementary tools for improving the performance of W-methods when applied to semi-discretized multidimensional systems of PDEs. Numerical experiments on representative advection–diffusion–reaction problems confirm the theoretical analysis and the robustness of the methods.

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On Low-Storage Runge-Kutta Schemes for Ordinary Differential Equations

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Abstract

In this talk, we study low-storage explicit RK methods: explicit Runge-Kutta (RK) schemes for the numerical solution of high-dimensional systems of ordinary differential equations (ODEs) that can be implemented using only two memory registers. We focus on fourth-order, four-stage explicit low-storage schemes of the Calvo type, which can be implemented with just two memory registers per step. In addition, we study five-stage, fifth-order methods for the numerical solution of linear inhomogeneous ODEs, as well as five-stages schemes that achieve fourth-order accuracy for general systems.
Finally, we evaluate the efficiency of these new methods by applying them to various differential systems and benchmarking them against traditional RK schemes and alternative low-storage models, including Zingg's methods, Ketcheson, Williamson and Van der Houwen ones.

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Nonstandard Finite Difference Methods: Structure Preservation, High Order, and Connections with SSP Theory

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Abstract

Production–destruction systems arise in a wide range of applications, including biological, epidemiological, and chemical models, where preserving qualitative properties such as positivity, conservation, and asymptotic stability is essential for reliable numerical simulation. In recent years, nonstandard finite difference (NSFD) methods have proven particularly effective in enforcing these structural features independently of the time-step size.
In this talk, we present recent advances in the construction and analysis of structure-preserving numerical integrators based on NSFD, Patankar, and Modified Patankar methodologies. We discuss a general framework for the derivation of positivity-preserving schemes of order higher than one, with particular emphasis on consistency improvements and the preservation of additional dynamical properties, including elementary stability and conservation laws. Particular attention will be given to new and ongoing work concerning the connection between NSFD/Patankar methods and strong stability preserving (SSP) theory. We present new theoretical insights suggesting a previously unexplored link between positivity-preserving nonstandard discretizations and SSP constructions. We also include numerical examples illustrating the behavior of the proposed approaches and highlights several open problems and future perspectives.

Novel approaches for the reliable and efficient numerical evaluation of Landau-type operators

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Abstract

Numerical approximations of Landau-type operators represent fundamental components of time integration methods for demanding problems such as inhomogeneous Vlasov-Landau-type equations. Substantial computational issues arise from the treatment of the physically most relevant three-dimensional case with Coulomb-type interaction.
Our recent work is concerned with the introduction and numerical comparison of novel approaches for the reliable and efficient evaluation of Landau-type collision operators, where the focus is on the treatment of integral operators involving general singular kernels. In the spirit of collocation, common tools are the identification of fundamental integrals, series expansions of the integral kernel and the density function on the main part of the velocity domain, and interpolation as well as quadrature approximation nearby the singularity of the kernel. Focusing on the favourable choice of the Fourier spectral method, their practical implementation uses the reduction to basic integrals, fast Fourier techniques, and summations along certain directions. Moreover, an important observation is that a significant percentage of the overall computational effort can be transferred to precomputations which are independent of the density function. For the purpose of exposition and numerical validation, the cases of constant, regular, and singular integral kernels are distinguished, and the procedure is adapted accordingly to the increasing complexity of the problem.

Landmarks in the History of Iterative Methods

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Abstract

Innehåll.
I. Ancient Square Root Calculations (Babylon, India, Egypt, Heron, Bakhshālī manuscript).
II. Fixed Point Iterations (Jamshīd Ghīāth ud-Dīn Al-Kāshī, Nityānanda, Ḥabash al-Ḥāsib al-Marwazī, J. Kepler, L. Euler, J. Fourier).
III. Emergence of Newton's Method (I. Newton, Chr. Wren, J. Wallis and Newton on Kepler's Equation, Th. Simpson).
IV. Geometry of Newton's Method (J. R. Mourraille, A. Cayley, G. Julia, J. Hubbard, A. Douady).
V. Error Estimates for Newton's Method (J. Fourier, A.-L. Cauchy , L. V. Kantorovich).

Efficient solution of singular BVPs in ODEs/DAEs

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Abstract

We deal with boundary value problems for systems of ordinary differential equations with singularities. Typically, such problems have the form

$$z'(t) = F(t, z(t)), t \in (0, 1], B_0 z(0) + B_1 z(1) = \beta, \alpha > 0,$$

where $\lim_{t\to 0} F(t, z(t)) = \infty$ and $\lim_{t\to 0} \partial F(t, z)/partial z = \infty$. The analysis is usually done for the model equation

$$z'(t) = \frac{1}{t^\alpha} M z(t) + f(t, z(t)), t \in (0, 1], B_0 z(0) + B_1 z(1) = \beta,$$

where $f(t, z)$ may also be in the form of $g(t, z)/t$ with a smooth function $g(t, z)$. We briefly recapitulate the analytical properties of the above problems with a special focus on the most general boundary conditions which guarantee their well-posedness.
To compute the numerical approximation for z we use polynomial collocation, because the method retains its hight order even in case of singularities. The usual high-order superconvergence at the mesh points does not hold in general. However, the uniform superconvergence is preserved (up to logarithmic factors).
The updated version of the Matlab code `bvpsuite2.0` with the special focus on the above problem class has been implemented. For higher efficiency, estimate of the global error and adaptive mesh selection are provided. The code can be applied to arbitrary order problems in implicit form. Also systems of index 1 differential-algebraic equations (DAEs) are in the scope of the code. We illustrate the performance of the software with a special focus on parameterdependent problems by means of numerical simulation of models in applications.

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