Maximum Flows and Minimum Cuts

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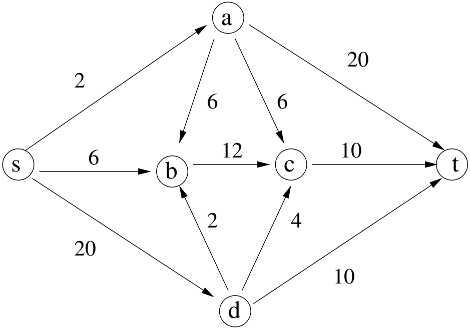
Minimum Cut Problem

The input:

- A directed graph $G = (V, E)$
- Two special vertices: source s and target t
- A function $c : E \to \mathbb{R}_{\geq 0}$, that assigns a capacity to each edge.

Minimum Cut Problem



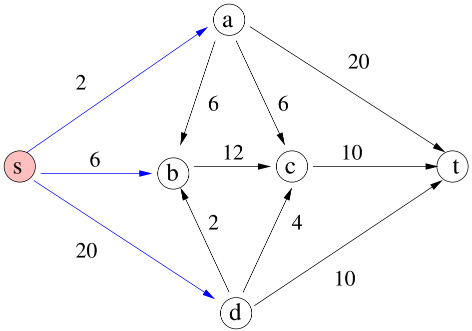
Minimum Cut Problem

- A (s,t)-cut is an ordered pair (S,T) such that
  1. S and T partition V
  2. $s \in S$ and $t \in T$
- $\text{Cap}(S,T)$, the capacity of cut (S,T), is the sum of the capacities of all edges crossing the cut (from S to T):

$$\sum_{u \in S, v \in T} c(u,v)$$

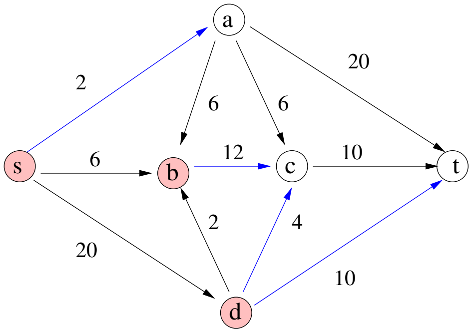
For notational convenience, we pretend $c(u,v) = 0$ if $(u,v) \notin E$.

Cut: Example 1



Capacity = 2 + 6 + 20 = 28

Cut: Example 2



Capacity = 2 + 12 + 4 + 10 = 28

Minimum Cut Problem

- The goal is to find an (s, t)-cut with minimum capacity
- Such a cut is known as the minimum capacity cut, or a minimum cut
- Removing edges crossing a minimum cut gives cheapest way of ensuring s cannot reach t.

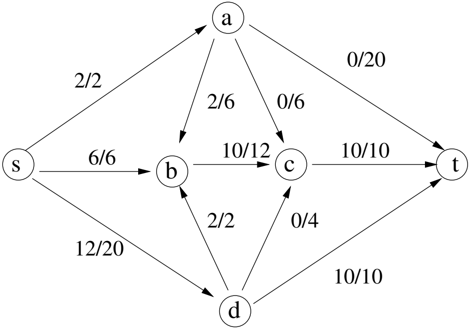
Maximum Flow Problem

- Input same as in minimum cut problem
- An (s, t)-flow is a function $f : E \to \mathbb{R}_{\geq 0}$ that satisfies the following conservation constraint at each vertex v other than the source s and sink t:

$$\sum_u f(u, v) = \sum_w f(v, w)$$

- That is, the total flow into v is the total flow out of v.
- $f(u, v)$ is referred to as the flow on (u, v).

Flow Example



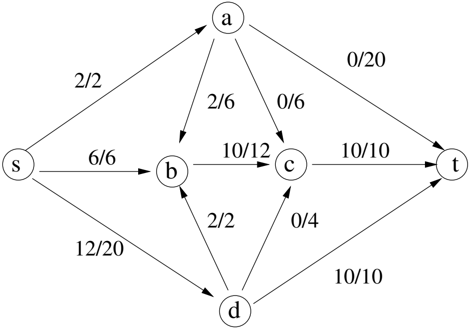
Value of Flow

- The value of flow f, denoted $|f|$, is

$$|f| := \sum_{w} f(s, w) - \sum_{u} f(u, s).$$

- That is, $|f|$ is the net flow out of the source s.

Flow Example



Value of flow = 2 + 6 + 12 = 20

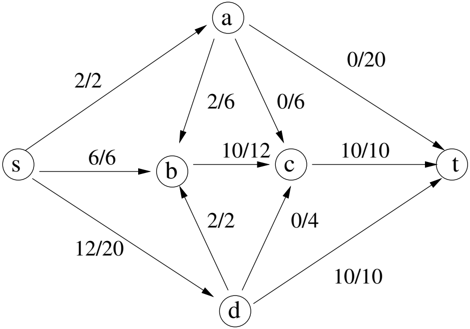
Feasible Flow

- A flow f is feasible if for each edge (u, v), flow on edge is at most its capacity:
$$f(u, v) \leq c(u, v)$$
- This is called the capacity constraint

Maximum Flow Problem

- The input is directed graph G along with edge capacities, source s and sink t.
- The goal is to find a feasible flow f that maximizes the value $|f|$. Such a maximizing flow is called a maximum flow

Flow Example



Value of flow = 2 + 6 + 12 = 20

Source vs. Sink

Let us use some notation for the net flow out of v:

$$\partial f(v) := \sum_w f(v,w) - \sum_u f(u,v)$$

We have:

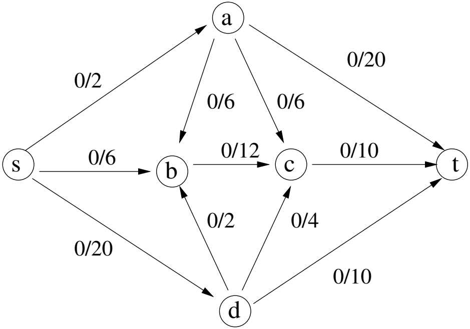
$$0 = \sum_v \partial f(v) = \partial f(s) + \partial f(t)$$

- The first equality follows because any edge (u,v) contributes $f(u,v)$ to $\partial f(u)$ and $-f(u,v)$ to $\partial f(v)$.
- The second equality uses flow conservation.

Source vs. Sink

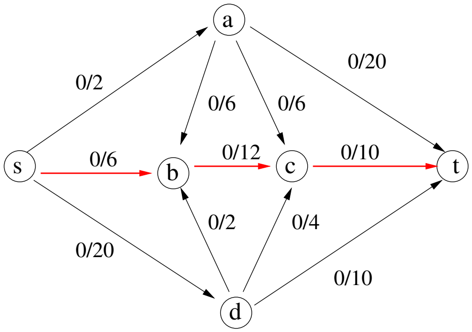
- Thus, $|f| = \partial f(s) = -\partial f(t)$
- That is, net flow out of s equals net flow into t. We will see a generalization of this idea a bit later.

Towards a Maximum Flow Algorithm



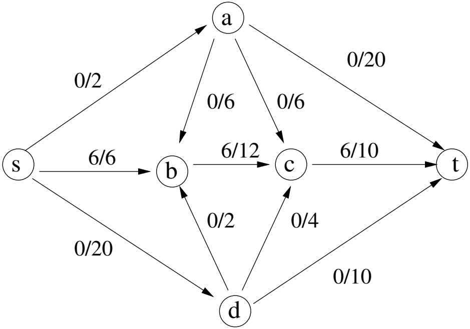
Value of flow = 0

Towards a Maximum Flow Algorithm



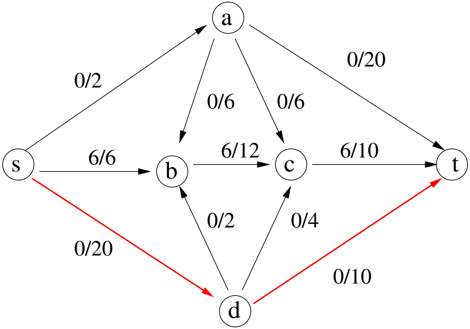
Value of flow = 0

Towards a Maximum Flow Algorithm



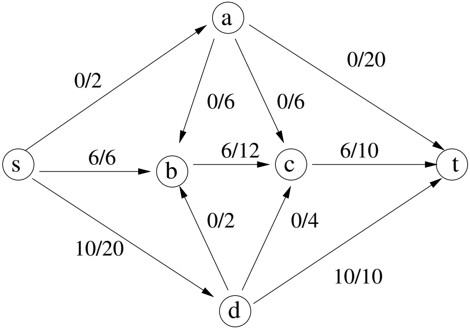
Value of flow = 6

Towards a Maximum Flow Algorithm



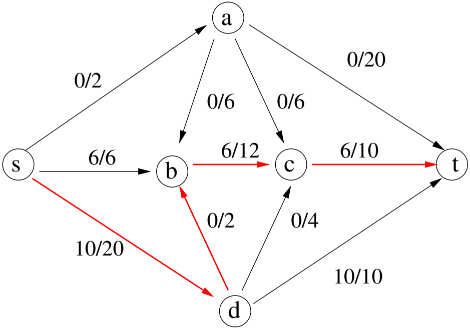
Value of flow = 6

Towards a Maximum Flow Algorithm



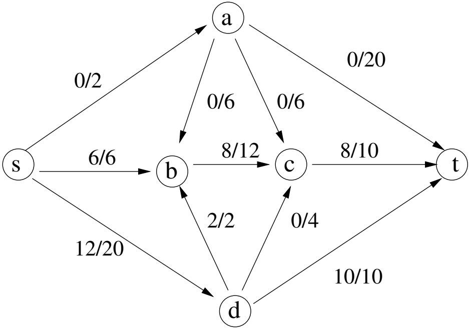
Value of flow = 16

Towards a Maximum Flow Algorithm



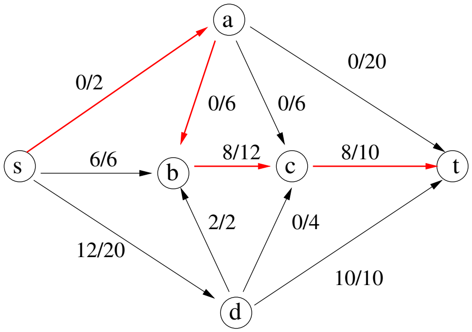
Value of flow = 16

Towards a Maximum Flow Algorithm



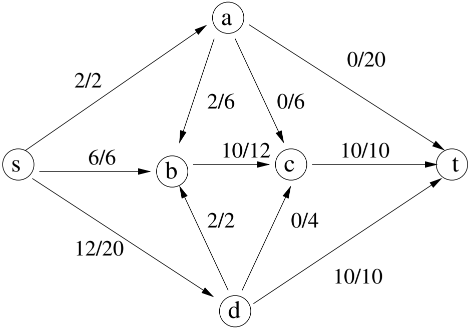
Value of flow = 18

Towards a Maximum Flow Algorithm



Value of flow = 18

Can we improve on this flow?



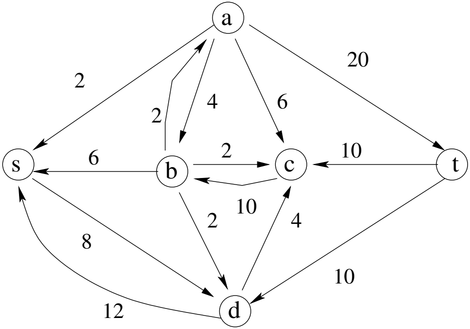
Value of flow = 20

Residual Network

Suppose we have a flow f that is feasible. Also assume that in the network G, for every pair of vertices u and v, at most one of (u, v) and (v, u) is an edge in G.

- For each $(u, v) \in E$, let
  1. $c_f(u, v) := c(u, v) - f(u, v)$
  2. $c_f(v, u) := f(u, v)$
- We call $c_f(e)$ the residual capacity of e.
- The residual network G_f consists of all edges whose residual capacity is strictly positive.

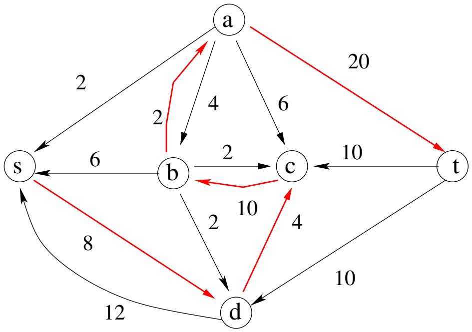
Residual Network



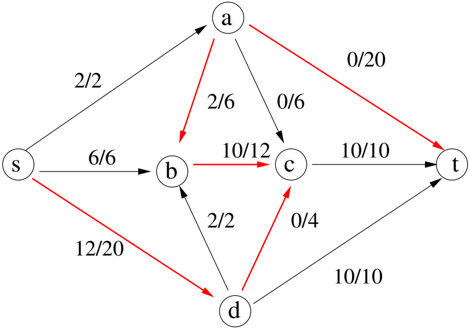
Ford-Fulkerson Algorithm

- Initialize f to the zero flow
- While (there is a path from s to t in residual G_f)
  - Let π be any (simple) path from s to t in G_f.
  - Augment f using π, as described below.

Flow Augmentation

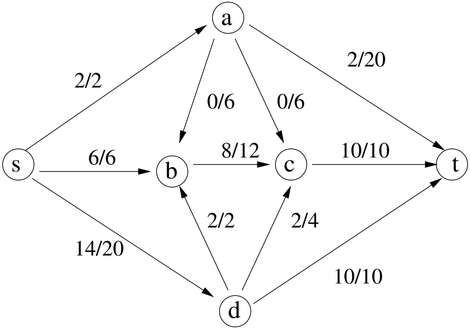


Flow Augmentation



Value of flow = 20

Augmented Flow



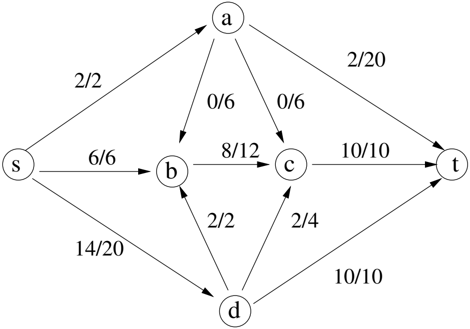
Value of flow = 22

Flow Augmentation

Augmenting flow f using path π in residual network G_f:

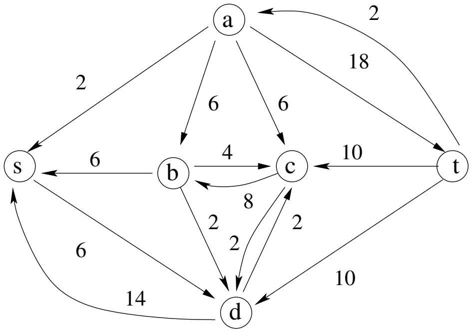
- Let $\mu > 0$ be the minimum residual capacity of an edge in π.
- For each edge $(u, v) \in G$
  - $f'(u, v) \leftarrow f(u, v) + \mu$ if (u, v) is on π
  - $f'(u, v) \leftarrow f(u, v) - \mu$ if (v, u) is on π.
  - $f'(u, v) \leftarrow f(u, v)$ otherwise.
- $f \leftarrow f'$.

Is this flow optimal?

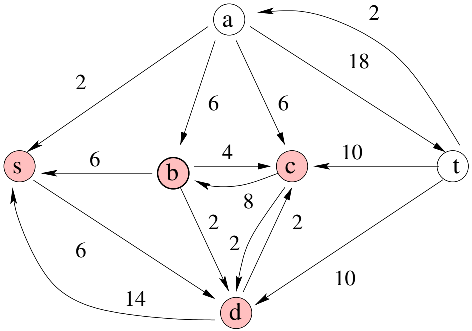


Value of flow = 22

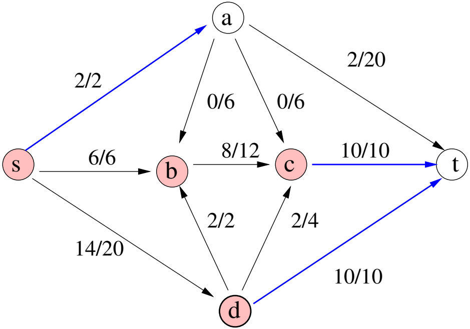
Algorithm Terminates: No path in residual network



Algorithm Terminates: Reachability in residual network



Flow vs. Cut



flow value = 22, cut capacity = 22