1. Prove the triangle inequality for point sets in \( \mathbb{R}^n \) endowed with the Euclidean norm. That is, show that for any three points \( p, q, r \in \mathbb{R}^n \),

\[
\|p - r\| \leq \|p - q\| + \|q - r\|.
\]

Hint: Use Cauchy-Schwartz inequality. Produce a proof of Cauchy-Schwartz either by yourself or by looking it up.

2. Prove the Brunn-Minkowski inequality for a pair of bricks. That is, let \( y_1, \ldots, y_n \geq 0 \) and \( z_1, \ldots, z_n \geq 0 \) be real numbers. Show that

\[
\left( \prod_{1 \leq i \leq n} y_i \right)^{1/n} + \left( \prod_{1 \leq i \leq n} z_i \right)^{1/n} \leq \left( \prod_{1 \leq i \leq n} (y_i + z_i) \right)^{1/n}
\]

Hint: One way to show this is to use a generalized version of the inequality that the geometric mean is bounded by the arithmetic mean.

3. Show that the local search algorithm for \( k \)-median terminates in polynomial time. If you do not have the notes for this algorithm, look up the paper but assume that the initial solution is the one produced by our algorithm for the \( k \)-center problem.

4. Let \( \text{Opt}(P, X) \) denote the value of the optimum solution for the \( k \)-means problem for a set of points \( P \) in metric space \( X \). Prove that for a set \( P \subseteq \mathbb{R}^n \), \( \text{Opt}(P, P) \leq 4\text{Opt}(P, \mathbb{R}^n) \).

5. Given an instance \( P \subseteq \mathbb{R}^n \) of \( n \) points for the \( k \)-means problem and an \( \varepsilon > 0 \), check that we can compute, in \( n^{O(1/\varepsilon^2)} \) time a subset \( X \subseteq \mathbb{R}^n \) with \( n^{O(1/\varepsilon^2)} \) points such that \( \text{Opt}(P, X) \leq (1 + \varepsilon)\text{Opt}(P, \mathbb{R}^n) \). Hint: We need to proceed along the lines of our proof of a similar result for \( k \)-center. You may want to read the paper by Badoiu et al. that implies such a result for the \( k \)-median. I use the word “check” rather than “prove” because I am not completely certain that the claim is true.