Market Equilibrium for CES Exchange Economies: Existence, Multiplicity, and Computation*

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Abstract. We consider exchange economies where the traders' preferences are expressed in terms of the extensively used *constant elasticity of substitution* (CES) utility functions. We show that for any such economy it is possible to say in polynomial time whether an equilibrium exists.
We then describe a convex formulation of the equilibrium conditions, which leads to polynomial time algorithms for a wide range of the parameter defining the CES utility functions. This range includes instances that do not satisfy weak gross substitutability. As a byproduct of our work, we prove the uniqueness of equilibrium in an interesting setting where such a result was not known.
The range for which we do not obtain polynomial-time algorithms coincides with the range for which the economies admit multiple disconnected equilibria.

1 Introduction

An *exchange economy* consists of a collection of goods, initially distributed among a number of traders. The preferences of the traders for the bundles of goods are expressed by a utility function. Each trader wants to maximize her utility, subject to her budget constraint.

An equilibrium is a set of prices at which there are allocations of goods to traders such that two conditions are simultaneously satisfied: each trader's allocation maximizes her utility subject to the budget constraint, and the market clears.

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Existence. An early and fundamental triumph of Mathematical Economics was the 1954 result by Arrow and Debreu [1] that, even in a more general situation which includes the production of goods, subject to mild sufficient conditions, there is an equilibrium. However, given a set of traders, each endowed with a concave utility function and a nonnegative vector of initial endowments, an equilibrium does not need to exist.

Thus the problem arises of determining whether a given exchange economy has an equilibrium. In this paper, we show that this problem can be solved in polynomial time, whenever the utility functions are of the form $u(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} c_j x_j^{\rho}\right)^{\frac{1}{\rho}}$, for $-\infty < \rho < 1$ and $\rho \neq 0$, i.e., for constant elasticity of substitution (CES) utility functions [27].

This result generalizes methods of Gale [13], who analyzed the existence of equilibria for linear utility functions, and Eaves [12], who analyzed the existence of positive equilibrium prices for Cobb-Douglas utility functions. (See also Jain [17], who employs a sufficient condition for the existence of positive price equilibria for linear utility functions.) Our result is in contrast with the NP-hardness result of [7], which applies to Leontief utility functions. As described below, linear, Cobb-Douglas, and Leontief utility functions are limiting cases of CES utility functions.

Computation. The problem of computing equilibrium prices for exchange economies has attracted a lot of attention since the 1960s. In recent years, theoretical computer scientists have become interested in the polynomial-time solvability of the problem. Several results [25] seem to indicate that in order for the problem to admit polynomial time algorithms, certain restrictions should be satisfied by the market.

Two well studied restrictions are *gross substitutability* – GS (see [22], p. 611) and the *weak axiom of revealed preferences* – WARP (see [22], Section 2.F). Although restrictive, these conditions are useful and model some realistic scenarios. A utility function satisfies GS (resp., weak GS – WGS) if increasing the prices of some of the goods while keeping the other prices and the income fixed causes the increase (resp., does not cause the decrease) in demand for the goods whose price is fixed. Roughly speaking, WARP means that the aggregate behavior of the market fulfills a fundamental property satisfied by the choices made by any rational individual trader.

It is well known that GS implies that the equilibrium prices are unique up to scaling ([28], p. 395), and that WGS and WARP both imply that the set of equilibrium prices is convex ([22], p. 608). When the set of equilibria is convex, it is enough to add a non-degeneracy assumption (which is almost always satisfied) to get the uniqueness of the equilibrium up to scaling [9].

Most of the polynomial-time algorithms developed so far apply to exchange economies where either WGS or WARP hold. In this paper we present a convex characterization of the equilibrium conditions which applies to exchange economies with CES functions such that $-1 \leq \rho < 0$. Note that these economies do not fall into either WGS or WARP. Also, the methods of [24, 17], which work when each utility function $u(x_1, \dots, x_n)$ has the property that $\log(\frac{u(x)}{\partial u(x)/\partial x_j})$ is a concave function for every j, do not apply here.

Multiplicity. Besides its algorithmic contribution, our work allows us to conclude that, for CES functions with $-1 \leq \rho < 0$, the equilibria are connected, and are thus essentially unique. This was not known by economists. Indeed it turns out that an exchange economy with traders endowed with CES utility functions such that $-1 \leq \rho < 0$ is not covered by any of the known conditions that ensure that there are no multiple disconnected equilibria, such as the *Super Cobb-Douglas Property* of Mas-Colell [21], and thus our result also provides an original contribution to the theory of equilibrium. Combined with a result by Gjerstad [16], who showed that multiple disconnected equilibria can arise in economies where traders have CES functions with any $\rho < -1$, our work leads to a characterization of the values of ρ for which the CES exchange economies equilibrium sets must be connected.

Related Work. In a series of papers which started with linear utility functions, polynomial time algorithms have been developed to compute equilibria for more and more general settings [10, 11, 18, 14, 17, 29, 3, 15, 6, 4]. However, the corresponding market satisfies one of the two conditions discussed above (WGS or WARP) (see [5] for a review).

The technical tool used in some of these results is to reformulate the problem in terms of mathematical programming in a way that a polynomial time algorithm (or approximation scheme – in general the equilibrium point is not a vector of rationals) can be obtained by known optimization techniques. In particular, *convex programming* has been proven to be a particularly useful tool [24, 17, 29, 6, 4].

Organization. In Section 2, we formally describe the model of an exchange economy, introduce CES functions, and hint at their economic relevance. Section 3 is devoted to a detailed discussion of the demand function of traders with CES utility functions. In Section 4, we characterize the problem of existence of an equilibrium for CES exchange economies, in terms of a graph property that can be verified in polynomial time. In Section 5 we show that equilibrium prices and allocations for an exchange economy, where the traders are endowed with CES functions with $-1 \leq \rho < 0$, can be computed by solving a feasibility problem, defined in terms of explicitly given convex constraints.

2 Background

We now describe the exchange market model. Let us consider m economic agents who represent traders of n goods. Let $\mathbf{R}^n_+$ (resp. $\mathbf{R}^n_{++}$) denote the subset of $\mathbf{R}^n$ where the coordinates are nonnegative (resp. strictly positive). The j-th coordinate in $\mathbf{R}^n$ will stand for the good j. Each trader i has a concave utility function $u_i : \mathbf{R}^n_+ \rightarrow \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \dots, w_{in}) \in \mathbf{R}^n_+$. At given prices $\pi \in \mathbf{R}^n_+$, the i-th trader will sell her endowment, and get the bundle of goods $x_i = (x_{i1}, \dots, x_{in}) \in \mathbf{R}^n_+$ which maximizes $u_i(x)$ subject to the budget constraint[4] $\pi \cdot x \leq \pi \cdot w_i$. The budget constraint restricts the choice to bundles that cost no more than $\pi \cdot w_i$, the *income* of trader i. If the utility maximization is well-defined, such a bundle x_i is called the *demand* of trader i at price π, and is denoted by $x_i(\pi)$. If the utility has no maximum over the set of feasible bundles, we say that the demand is not well-defined. (The feasible region is always non-empty, since the origin is in it.)

An equilibrium is a nonnegative vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ of goods for each trader i such that the following two conditions hold:

1. For each trader i, the demand is well-defined at price π and $\bar{x}_i$ is a demanded bundle.
2. For each good j, $\sum_i \bar{x}_{ij} \leq \sum_i w_{ij}$.

Under the assumption that for each i and every bundle $x \in \mathbf{R}^n_+$, there is a bundle $y \in \mathbf{R}^n_+$ such that $u_i(y) > u_i(x)$, it can be shown that for any good with positive price, equality must hold in (2). The already mentioned result of Arrow and Debreu [1] implies that, under some mild assumptions, such an equilibrium exists. The above described market model is usually called an *exchange economy*.

A special (and analytically more tractable) case of the exchange model, known as Fisher's model, arises when the economic agents are buyers, endowed with fixed incomes, competing for goods, which are available in fixed quantities. Note that Fisher's model can be seen as a special case of an exchange economy, obtained by assuming that the initial endowments are *proportional*, i.e., $w_i = \delta_i w$, $\delta_i > 0$, so that the relative incomes of the traders are independent of the prices.

CES utility functions. The most popular family of utility functions is given by CES (constant elasticity of substitution) functions, which have been introduced in [27]. We refer the reader to the book by Shoven and Whalley [26] for a sense of their pervasiveness in applied general equilibrium models. A CES function ranks the trader's preferences over bundles of goods $(x_1, \ldots, x_n)$ according to the value of $u(x_1, \ldots, x_n) = \left(\sum_{j=1}^n c_j x_j^\rho\right)^{\frac{1}{\rho}}$. where $-\infty < \rho < 1$, but $\rho \neq 0$.

The success of CES functions is due to the useful combination of their mathematical tractability with their expressive power, which allows for a realistic modeling of a wide range of consumer preferences. Indeed, one can model markets with very different characteristics in terms of preference towards variety, substitutability versus complementarity, and multiplicity of price equilibria, by changing the values of ρ and of the utility parameters c_j.

CES functions have been thoroughly analyzed in [2], where it has also been shown how to derive, in the limit, their special cases, i.e., linear, Cobb-Douglas, and Leontief functions (see [2], p. 231). Let $\sigma = \frac{1}{1-\rho}$. The parameter σ is called the *elasticity of substitution*. For $\sigma \to \infty$ ($\rho \to 1$), CES take the linear form, and the goods are perfect substitutes, so that there is no preference for variety. For

[4] Given two vectors x and y, we use the notation $x \cdot y$ to denote their inner product.

$\sigma > 1$ $(\rho > 0)$, the goods are partial substitutes, and different values of σ in this range allow us to express different levels of preference for variety. For $\sigma \to 1$ $(\rho \to 0)$, CES become Cobb-Douglas functions, and express a perfect balance between substitution and complementarity effects. Indeed it is not difficult to show that a trader with a Cobb-Douglas utility spends a fixed fraction of her income on each good.

For $\sigma < 1$ $(\rho < 0)$, CES functions model markets with significant complementarity effects between goods. This feature reaches its extreme (*perfect complementarity*) as $\sigma \to 0$ $(\rho \to -\infty)$, i.e., when CES takes the form of Leontief functions. In the latter case, the *shape* of the optimal bundle demanded by the consumer does not depend at all on the prices of the goods, but is fully determined by the parameters defining the utility function.

Whenever the relative incomes of the traders are independent of the prices, CES functions give rise to a market which satisfies WARP. This happens for instance in the Fisher model, a very special case of the exchange model. On the other hand, CES functions satisfy WGS if and only if $\rho \geq 0$, whereas, if $\rho < -1$, they allow for multiple disconnected equilibria.

3 Demand of CES Consumers

In this section, we characterize the demand function of traders with CES utility functions. Consider a setting where trader i has an initial endowment $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}_+^n$ of goods, and the CES utility function $u_i(x_{i1}, \ldots, x_{in}) = \left(\sum_{j=1}^n \alpha_{ij} x_{ij}^{\rho_i}\right)^{\frac{1}{\rho_i}}$, where $\alpha_{ij} \geq 0$, and $-\infty < \rho_i < 1$, but $\rho_i \neq 0$.

We assume throughout that each trader i *owns* some good j, that is, $w_{ij} > 0$ for some j. We also assume that each trader i *wants* some good j, that is, $\alpha_{ij} > 0$ for some j. If trader i does not want good j, it is easy to see that the utility of a bundle $x_i \in \mathbf{R}_+^n$ is independent of x_{ij}. We adopt the convention that $\alpha_{ij} x_{ij}^{\rho_i} = 0$ when $\alpha_{ij} = 0$ and $x_{ij} = 0$. If $\rho_i < 0$, we define $u_i(x_{i1}, \ldots, x_{in}) = 0$ if there is a j such that i wants j and $x_{ij} = 0$. Note that this ensures that u_i is continuous over $\mathbf{R}_+^n$.

Consider a case where $\rho_i > 0$. Evidently, if we start with any bundle $x_i \in \mathbf{R}_+^n$ and add to it an arbitrarily small amount of a good that i wants, we get a bundle with more utility. From this, it follows that the demand of the trader is well-defined at a given price if and only if each of the goods that the trader wants has a strictly positive price.

Now consider the case where $\rho_i < 0$. A bundle $x_i \in \mathbf{R}_+^n$ has a strictly positive utility if and only if it has a strictly positive amount of each of the goods that the trader wants. Evidently, if we start with any bundle $x_i \in \mathbf{R}_+^n$ that has strictly positive utility and add to it an arbitrarily small amount of a good that i wants, we get a bundle with more utility. Let π be a price at which the income $\pi \cdot w_i$ is positive. Since the trader can afford a bundle with positive utility, we conclude that the demand is well-defined at π if and only if each of the goods that the trader wants has a strictly positive price. Now let π be a price at which the income $\pi \cdot w_i$ is zero. We see that the demand is well-defined if and only if *at least one* of the goods that the trader wants is positively priced.

Irrespective of whether ρ_i is positive or negative, traders with positive income demand a positive amount of each good they want. Such positive income traders are also *non-satiable* on all goods they want which means that demand is not well-defined if any good they want is priced zero.

Also irrespective of whether ρ_i is positive or negative, the demand is well-defined at any strictly positive price vector $\pi \in \mathbf{R}^n_{++}$. It is in fact unique and is given by the expression

$$x_{ij}(\pi) = \frac{\alpha_{ij}^{1/1-\rho_i}}{\pi_j^{1/1-\rho_i}} \times \frac{\sum_k \pi_k w_{ik}}{\sum_k \alpha_k^{1/1-\rho_i} \pi_k^{-\rho_i/1-\rho_i}}. \tag{1}$$

The formula above is folklore and is derived using the Kuhn-Tucker conditions.

4 Existence of an Equilibrium

The celebrated paper of Arrow and Debreu [1] had a much weaker set of assumptions sufficient for the existence of equilibrium than earlier work. However, the assumptions were still somewhat restrictive. Indeed, Arrow and Debreu themselves called the assumptions for their first existence theorem "clearly unrealistic" and immediately proceeded to weaken the sufficent conditions for their second theorem. See the introduction to Maxfield [23] for a discussion of the work on showing existence of equilibrium under progressively weaker assumptions. In general, it is NP-hard to determine whether a market possesses an equilibrium or not [7].

Gale [13] provided a very simple two trader example of a market that does not possess an equilibrium. Gale's example was for the linear exchange model, but it also serves as an example for the CES case with $\rho > 0$. Suppose trader one possesses both apples and oranges, but only wants apples. Trader two wants both apples and oranges, but owns only oranges. This simple market has no equilibrium. If oranges are priced at zero, then the demand of trader two is not well-defined. If oranges have a positive price, then trader one will want to sell all of her oranges to buy more apples even though she already owns all the apples present in the market. Gale's example will not work for the CES with $\rho < 0$ case though because that actually has an equilibrium with a positive price for apples and zero price for oranges.

In this section, we characterize the existence of equilibrium for an exchange economy where the traders have CES utility functions. The characterization immediately implies a polynomial time algorithm to decide whether the economy has an equilibrium. As before, we assume that each trader wants at least one good and owns at least one good. We also assume that each good is owned by some trader.

We assume in the remainder of this section that each trader has a positive amount of precisely one good. This assumption is without loss of generality: we may replace a trader with positive amounts of k different goods with k traders, each with the same utility function and a positive amount of one good. A straightforward argument that employs the homogeneity of the CES utility functions shows that this transformation preserves the equilibria.

It is easy to see, but nonetheless worth noting, that the traders with positive income will be precisely those traders whose single good is positively priced.

Definition 1. *There is a vertex v_i for each consumer i. We have an arc from v_i to v_k when trader i possesses a good which trader k wants. The resulting directed graph is called an economy graph.*

The following existence theorem is the main result we use from Maxfield [23].

Theorem 1. *If the economy graph is strongly connected, an equilibrium exists. Moreover, all goods are positively priced at any equilibrium.*

Proof. This follows from Theorem 2 of Maxfield [23] who obtains this result using strong connectivity and general results on the existence of a *quasi-equilibrium* ([22], Chapter 17). □

Definition 2. *We say that a strongly connected component is on (at a given price) if every trader within it has a positive income. If no trader in a strongly connected component has a positive income, then we say that that component is off.*

Lemma 1. *At equilibrium, every strongly connected component in an economy graph is either on or off.*

Proof. Suppose not. Suppose that at equilibrium price π, there is a component that is neither on or off. In that case, there must be a trader with positive income that desires a good from a trader with no income. That means the zero income trader's good must have a price of zero. Since the trader with positive income is non-satiable on the zero priced good, demand is not well-defined for that good and therefore, π is not an equilibrium. This provides a contradiction. □

Consider a strongly connected component C of the economy graph that has no incoming arcs from traders outside C. We claim that a good held by any trader i in C is also desired by some trader i' in C. If C consists simply of the node v_i, then since there are no incoming arcs from outside, it must be that i desires his own good. If C consists of more than one node, the claim follows from strong connectivity.

Furthermore, it now follows that a good held by a trader in C is not held by any trader outside C. Otherwise, C would have an incoming arc.

Lemma 2. *At equilibrium, a strongly connected component of an economy graph is on if and only if it has no incoming arcs.*

Proof. Suppose the economy has an equilibrium price π. Suppose a strongly connected component C_1 is on. We will show that C_1 can have no incoming arcs. If C_1 has an incoming arc, that means some trader t in C_1, wants some good g held by a trader in another component C_2. If C_2 is off, then g has price zero. But t has positive income since C_1 is on. Since t wants g, t's demand is undefined, thus contradicting the assumption of equilibrium. Thus, at equlibrium C_2 must be on. If C_2 has any incoming arcs, then we can make an identical argument to show that the components providing the incoming arcs must also be on. Following this chain, we arrive at two components C'_1 and C'_2 that are on, C'_1 has an incoming arc from a trader in C'_2, and C'_2 has no incoming arcs. So a trader t in C'_1, who has positive income, will demand a positive amount of a good g that is held by some trader in C'_2. Since C'_2 has no incoming arcs, g is owned only by traders in C'_2, as we have already established. Since C'_2 has no incoming arcs, the traders in C'_2 form a subeconomy for which π is seen to be an equilibrium. Since the price of all goods held by traders in C'_2 is positive (C'_2 is on), it holds that for all such goods, including g, the demand within C'_2 equals the supply within C'_2. But this means that the demand for g in the bigger economy exceeds the supply: t, who is outside C'_2, demands a positive amount of it, but only traders in C'_2 own it. Thus, π is not an equilibrium which is a contradiction.

Suppose the economy has an equilibrium price π. Suppose further that a component, C, has no incoming arcs. We show that C must be on. Suppose C is off. Consider any trader in C. He wants some goods; all of these are owned only by traders in C, since C has no incoming arc. All goods in C are free (C is off), so the trader's demand is undefined. Therefore, π is not an equilibrium. We have a contradiction and the lemma is proven. $\square$

There is an important distinction, which bears repeating, between CES utility functions with $\rho > 0$ and those with $\rho < 0$. Traders with $\rho > 0$ will have positive utility as long as they have a positive amount of some good that they desire. Traders with $\rho < 0$ will only have positive utility if they have a positive amount of all goods they desire. Moreover, traders with $\rho > 0$ with zero income have undefined demands if any of their desired goods are priced at zero. Zero income traders with $\rho < 0$ only have undefined demand if all of their desired goods are free.

The following theorem is the main result of this section.

Theorem 2. *An equilibrium exists if and only if for every vertex v in a strongly connected component with incoming arcs, either (a) v has a CES utility function with $\rho > 0$ and all its incoming arcs are from vertices in strongly connected components without incoming arcs, or (b) v has a CES utility function with $\rho < 0$ and has at least one incoming arc from a strongly connected component without incoming arcs.*

Proof. Suppose an equilibrium price π exists. Then by Lemma 2, the strongly connected components that are on are precisely those that have no incoming arcs. And it is precisely the goods that are held by traders in such components that have positive price. Let C_1 be a strongly connected component with incoming arcs (if none exist, then this direction of the theorem is trivially true). Suppose there is a vertex i with a CES utility function with $\rho > 0$, and it has an incoming arc from a vertex that is in a strongly connected component with incoming arcs. Then i wants a good with price zero and so her demand is not defined, contradicting the assumption that π is an equilibrium price. Now suppose that there is a vertex i with a CES utility function with $\rho < 0$, and none of its incoming arcs are from a vertex in a strongly connected component with no incoming arcs. This means that trader i desires only zero priced goods and thus has undefined demand contradicting the assumption that π is an equilibrium price.

We now establish the other direction of the theorem. Each strongly connected component with no incoming arcs can be considered as an economy unto itself, and has an equilibrium with positive prices by Maxfield's theorem. For each good in a component with no incoming arcs we assign a price identical to its equilibrium price in the subeconomy. As no good in one of these strongly connected components is owned outside the component, this assignment of prices is well-defined.

For each good held by a trader in a component with incoming arcs, we assign a price of zero. By the argument above, we know that none of these goods are the same as those that were priced positively so this assignment is well defined. We claim that this price π is an equilibrium price.

For a trader in a component without incoming arcs, we assign the bundle that is the same as the one she gets in the equilibrium for the corresponding subeconomy. Clearly, this is a valid demand.

Consider a trader in a component with incoming arcs. Her income is 0. We claim that her demand is well-defined and that the zero bundle is a valid demand vector. This is because she is either a CES trader with $\rho > 0$ and all the goods that she wants are in components with no incoming arcs and hence positively priced, or she is a CES trader with $\rho < 0$ and at least one of the goods that she wants is positively priced, and thus the best utility she can afford is 0.

We now verify that condition (2) in the definition of an equilibrium holds, that is, the demand is at most the supply. For a good held by a trader in a component with no incoming arc, this follows from the equilibrium conditions of the corresponding subeconomy, and the fact that any trader outside the subeconomy demands 0 units of the good. For a good held by a trader in a component with incoming arcs, the net demand is 0, so condition (2) trivially holds. $\square$

We conclude by noting that besides yielding a polynomial time algorithm for checking the existence of equilibrium, the above characterization provides a polynomial-time reduction of the computation of an equilibrium for the original economy to the computation of positive price equilibria for sub-economies.

5 Efficient Computation by Convex Programming

In this section, we consider an economy in which each trader i has a CES utility function with $-1 \leq \rho_i < 0$. We show that the positive price equilibria of such an economy can be characterized as the solutions of a convex feasibility problem. The results of the previous section show that the computation of an equilibrium for an economy can be reduced to the computation of a positive price equilibrium for a sub-economy. This reduction, together with the fact that the convex feasibility problem can be solved (approximately) in polynomial time lead to a polynomial time algorithm for computing an approximate equilibrium. The notion of approximate equilibrium that we use corresponds to the strong approximate equilibrium defined by Codenotti et al.[6]; here, the condition (2) in the definition of an equilibrium is relaxed so that it holds approximately. Our algorithm will be polynomial not only in the input parameters but also in the number of bits used in the standard encoding of the rational number representing the approximation parameter. (We postpone a detailed discussion of this to a fuller version.) Whenever the solution can be irrational, such an algorithm is considered equivalent to an exact algorithm.

Since the demand of every trader is well-defined and unique at any positive price, we may write the positive price equilibria as the set $\pi \in \mathbf{R}_{++}$ such that for each good j, we have $\sum_i x_{ij}(\pi) \le \sum_i w_{ij}$. Let $\rho = -1$, and note that $\rho \le \rho_i$, for each i. Let $f_{ij}(\pi) = \pi_j^{1/(1-\rho)} x_{ij}(\pi)$. Let $\sigma_j = \pi_j^{1/(1-\rho)}$. In terms of the σ_j's, we obtain the set of $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}_{++}$ such that for each good j,

$$\sum_i f_{ij}(\sigma) \le \sigma_j (\sum_i w_{ij}).$$

We argue that this is a convex feasibility program. Since the right hand side of each inequality is a linear function, it suffices to argue that the left hand side is a convex function. The latter is established via the following proposition.

Proposition 1. *The function $f_{ij}(\sigma)$ is a convex function over $\mathbf{R}_{++}$.*

Proof. If $\alpha_{ij} = 0$, f_{ij} is zero over the domain and the proposition follows. Otherwise, f_{ij} is positive at each point of the domian. It therefore suffices to show that the constraint $f_{ij} \le t$ defines a convex set for positive t. Using the formula (1) for the demand, this constraint is

$$\frac{\alpha_{ij}^{\frac{1}{1-\rho_i}}}{\sigma_j^{\frac{\rho_i-\rho}{1-\rho_i}}} \times \frac{\sum_k \sigma_k^{1-\rho} w_{ik}}{\sum_k \alpha_{ik}^{\frac{1}{1-\rho_i}} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}}} \le t.$$

Rewriting, and raising both sides to the power $1/(1-\rho)$, we obtain

$$\alpha_{ij}^{\frac{1}{(1-\rho)(1-\rho_i)}} \times (\sum_k \sigma_k^{1-\rho} w_{ik})^{\frac{1}{1-\rho}} \le t^{\frac{1}{1-\rho}} \sigma_j^{\frac{\rho_i-\rho}{(1-\rho_i)(1-\rho)}} v_i^{\frac{-\rho_i}{1-\rho_i}}, \tag{2}$$

where

$$v_i = \left(\sum_k \alpha_{ik}^{\frac{1}{1-\rho_i}} \sigma_k^{\frac{-\rho_i(1-\rho)}{1-\rho_i}} \right)^{\frac{1-\rho_i}{-\rho_i(1-\rho)}}. \tag{3}$$

The left hand side of inequality 2 is a convex function, and the right hand side is a concave function that is non-decreasing in each argument when viewed as a function of t, σ_j, and v_i, since the exponents are non-negative and add up to one. Since $0 < \frac{-\rho_i(1-\rho)}{1-\rho_i} \leq 1$, the right hand side of equality 3 is a concave function, in fact a CES function. It follows that the right hand side of inequality 2 remains a concave function when v_i is replaced by the right hand side of equality 3. This completes the proof.

The convex feasibility formulation derived in this section highlights an independently useful property of the demand, encapsulated by Proposition 1. As we will show in a fuller version of this paper, a similar approach works for CES functions with $\rho > 0$, as well as for some other utility functions. The tools developed here for exchange economies also find some use in an extension to production [19].

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References

1. K.J. Arrow and G. Debreu, Existence of an Equilibrium for a Competitive Economy, Econometrica 22 (3), pp. 265–290 (1954).
2. K.J. Arrow, H.B. Chenery, B.S. Minhas, R.M. Solow, Capital-Labor Substitution and Economic Efficiency, The Review of Economics and Statistics, 43(3), 225–250 (1961).
3. N. Chen, X. Deng, X. Sun, and A. Yao, Fisher Equilibrium Price with a Class of Concave Utility Functions, ESA 2004.
4. B. Codenotti, B. McCune, K. Varadarajan, Market Equilibrium via the Excess Demand Function, STOC 2005.
5. B. Codenotti, S. Pemmaraju, K. Varadarajan, Algorithms Column: The Computation of Market Equilibria. SIGACT News Vol. 35(4), December 2004.
6. B. Codenotti, S. Pemmaraju, K. Varadarajan, On the Polynomial Time Computation of Equilibria for certain Exchange Economies. SODA 2005.
7. B. Codenotti, A. Saberi, K. Varadarajan, Y. Ye, Leontief Economies Encode Nonzero Sum Two-Player Games. Electronic Colloquium on Computational Complexity, Report TR05-055. To appear in SODA 06.
8. B. Codenotti, K. Varadarajan, Efficient Computation of Equilibrium Prices for Markets with Leontief Utilities, ICALP 2004.
9. G. Debreu, Economies with a Finite Set of Equilibria. Econometrica, vol. 38(3), pp. 387-92 (1970).
10. X. Deng, C. H. Papadimitriou, M. Safra, On the Complexity of Equilibria, STOC 02.
11. N. R. Devanur, C. H. Papadimitriou, A. Saberi, V. V. Vazirani, Market Equilibrium via a Primal-Dual-Type Algorithm. FOCS 2002, pp. 389-395.
12. B. C. Eaves, Finite Solution of Pure Trade Markets with Cobb-Douglas Utilities, Mathematical Programming Study 23, pp. 226-239 (1985).

13. D. Gale, The Linear Exchange Model, Journal of Mathematical Economics (3), 205–209, 1976.
14. R. Garg and S. Kapoor, Auction Algorithms for Market Equilibrium. In *Proc. STOC*, 2004.
15. R. Garg, S. Kapoor, and V. V. Vazirani, An Auction-Based Market Equilibrium Algorithm for the Separable Gross Substitutibility Case, APPROX 2004.
16. S. Gjerstad. Multiple Equilibria in Exchange Economies with Homothetic, Nearly Identical Preference, University of Minnesota, Center for Economic Research , Discussion Paper 288, 1996.
17. K. Jain, A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities, Proc. FOCS 2004.
18. K. Jain, M. Mahdian, and A. Saberi, Approximating Market Equilibria, Proc. APPROX 2003.
19. K. Jain and K. Varadarajan. Equilibria for Economies with Production: Constant-Returns Technologies and Production Planning Constraints. To appear in SODA 06.
20. O. L. Mangasarian. Nonlinear Programming, McGraw-Hill, 1969.
21. A. Mas-Colell, On the Uniqueness of Equilibrium Once Again, in: Equilibrium Theory and Applications, W. Barnett, B. Cornet, C. DAspremont, J. Gabszewicz, andA. Mas-Colell (eds). Cambridge University Press (1991).
22. A. Mas-Colell, M.D. Whinston, and J.R. Green, Microeconomic Theory, Oxford University Press (1995).
23. R. R. Maxfield, General Equilibrium and the Theory of Directed Graphs, Journal of Mathematical Economics 27, 23-51 (1997).
24. E. I. Nenakov and M. E. Primak. One algorithm for finding solutions of the Arrow-Debreu model, Kibernetica 3, 127–128 (1983). (In Russian.)
25. C.H. Papadimitriou, On the Complexity of the Parity Argument and other Inefficient Proofs of Existence, Journal of Computer and System Sciences 48, pp. 498-532 (1994).
26. J. B. Shoven and J. Whalley. Applying General Equilibrium, Cambridge University Press (1992).
27. R. Solov, A Contribution to the Theory of Economic Growth, Quarterly Journal of Economics 70, pp. 65-94 (1956).
28. H. Varian, Microeconomic Analysis, New York: W.W. Norton, 1992.
29. Y. Ye, A Path to the Arrow-Debreu Competitive Market Equilibrium. To appear in Mathematical Programming.