Addendum to "A Constant-Factor Approximation for Multi-Covering with Disks"

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In this note, we briefly outline a different view of the algorithm described in the paper "A Constant-Factor Approximation for Multi-Covering with Disks". This view generalizes and elucidates what that algorithm is actually doing, and in the process also resolves the first open problem posed in the paper, which is to obtain an $O(1)$ approximation for the problem when the objective function is the sum of the radii of the disks. An exposition that elaborates on this note is under preparation.

1 Computing a Covering for the Non-Uniform MCMC Problem

We solve the variant of the non-uniform MCMC problem where we have l_∞ disks rather than l_2 disks. Our input is two point sets Y and X in $\mathbb{R}^2$ and a coverage function $\kappa : X \to \mathbb{N} \cup \{0\}$. (It will be useful to allow $\kappa(x)$ to be 0 for some $x \in X$.) We also assume that $\kappa(x) \leq |Y|$ for each $x \in X$, for otherwise there is no feasible solution.

We describe an algorithm for assigning a *radius* $r_y \geq 0$ for each $y \in Y$, with the guarantee that for each $x \in X$, there are at least $\kappa(x)$ points $y \in Y$ such that the l_∞ disk of radius r_y centered at y contains x. In other words the guarantee is that for each $x \in X$,

$$|\{y \in Y \mid ||x - y||_\infty \leq r_y\}| \geq \kappa(x)$$

Our objective is to minimize the sum of the α-th powers of the radii of the disks, that is, $\sum_{y \in Y} r_y^\alpha$. Here we allow any $\alpha \geq 1$, whereas in the paper, we had $\alpha = 2$. (Our result can be stated in terms of a somewhat more general objective function, as a careful reader may observe.)

For this optimization problem, we will show that our algorithm outputs an $O(1)$ approximation. Clearly, this also gives an $O(1)$ approximation for the original problem, where distances are measured in the l_2 norm. We will follow the terminology as introduced in the paper.

1.1 Outer Cover

Given $X' \subseteq X$, Y, and κ, we need an auxiliary procedure OuterCover(X', Y, κ) that returns an assignment $\rho : Y \to \mathbb{R}^+$ of radii to the servers. The output returned must satisfy the condition that for each client $x \in X'$, there is a server $y \in Y$ such that (a) the disk $\delta(y, \rho_y)$ contains x, and (b) the radius $\rho_y \geq ||x - y^\kappa(x)||$. The procedure will compute an $O(1)$ approximation to the assignment that satisfies these conditions and minimizes $\sum_y \rho_y^\alpha$.

(For the sake of the reader's intuition, we note that when $\alpha = 2$, such an assignment can be obtained by taking the primary disks computed by the algorithm in the paper and expanding their radius by a factor of 3. In the present context, the output produced by OuterCover(X', Y, κ) will play the role that the primary disks did in the paper.)

The procedure OuterCover(X', Y, κ) can be implemented for any $\alpha \geq 1$ via a modification of the algorithm of Charikar and Panigrahy [7]. Note that their algorithm can be viewed as solving the case where $\kappa(x) = 1$ for each $x \in X'$. We can establish the $O(1)$ approximation guarantee for the procedure OuterCover(X', Y, κ) as well.

1.2 The Algorithm

With the procedure OuterCover(X', Y, κ) in place, we can now state our algorithm.

Algorithm 1 Cover(X, Y, κ)

if $\forall x \in X, \kappa(x) = 0$ **then**
    Assign $r_y \leftarrow 0$ for each $y \in Y$, and return.
Define $\kappa'(x)$ as follows:

$$\forall x \in X, \kappa'(x) = \begin{cases} 0, & \text{if } \kappa(x) = 0 \\ \kappa(x) - 1, & \text{if } \kappa(x) > 0 \end{cases}$$

Recursively call Cover(X, Y, κ').
Let $X' = \{x \in X \mid x \text{ is not } \kappa(x)\text{-covered }\}$
Call the procedure OuterCover(X', Y, κ) to obtain an assignment $\rho : Y \to \mathbb{R}^+$.
Let $Y' \leftarrow Y$.
while $X' \neq \emptyset$ **do**
    Choose $\overline{y} \in Y'$.
    Let $\text{XC}_{\overline{y}} \leftarrow \emptyset$, $\text{YC}_{\overline{y}} \leftarrow \emptyset$.
    for all $x' \in X'$ **do**
        if $x' \in \delta(\overline{y}, \rho_{\overline{y}})$ and $\rho_{\overline{y}} \geq ||x' - y^\kappa(x')||$ **then**
            $\text{XC}_{\overline{y}} \leftarrow \text{XC}_{\overline{y}} \cup \{x'\}$.
            $\text{YC}_{\overline{y}} \leftarrow \text{YC}_{\overline{y}} \cup \{y^1(x'), y^2(x'), \ldots, y^\kappa(x')\}$.
    Let $\text{YC}'_{\overline{y}} \subseteq \text{YC}_{\overline{y}}$ be a set of at most four points such that

$$\bigcap_{y \in \text{YC}'_{\overline{y}}} \delta(y, r_y) = \bigcap_{y \in \text{YC}_{\overline{y}}} \delta(y, r_y).$$

    For each $y \in \text{YC}'_{\overline{x}}$, increase r_y by the smallest amount that ensures $\text{XC}_{\overline{y}} \subseteq \delta(y, r_y)$.
    Remove $\overline{y}$ from Y and remove from X' any points x that are $\kappa(x)$-covered.

The algorithm computes an assignment of radius r_y to each server $y \in Y$ such that each client $x \in X$ is $\kappa(x)$-covered. This follows via arguments similar to those in the paper.

1.3 Approximation Ratio

To establish the approximation ratio, we need the following observation, which is the equivalent of Lemma 1 of the paper.

Lemma 1. *The increase in the objective function $\sum_{y \in Y} r_y^\alpha$ from the time Cover(X, Y, κ') completes to the time Cover(X, Y, κ) completes is $O(\sum_{y \in Y} \rho_y^\alpha)$.*

We can then bound the approximation ratio of the algorithm.

Theorem 1. *Let $r' : Y \to \mathbb{R}^+$ be any assignment of radii to the points in Y under which each point $x \in X$ is $\kappa(x)$-covered. Then the cost of the output of Cover(X, Y, κ) is at most $c * \sum_{y \in Y} {r'_y}^\alpha$, where $c > 0$ is an absolute constant.*

Proof. Our proof is by induction on $\max_{x \in X} \kappa(x)$. For the base case, where $\kappa(x) = 0$ for each $x \in X$, the claim in the theorem clearly holds for any $c > 0$.

Let $D = \{\delta(y, r'_y) \mid y \in Y\}$ be the set of disks corresponding to the assignment r'. Our proof strategy is to show that there is a subset $D_\kappa \subseteq D$ such that

1. The cost increase incurred by Cover(X, Y, κ) in going from the κ'-cover to the κ-cover is at most c times the sum of the α-th powers of the radii of the disks in D_k.

2. The set of disks $D \setminus D_\kappa$ $\kappa'(x)$-covers any point $x \in X$.

By the induction hypothesis, the cost of the κ'-cover computed by Cover(X, Y, κ') is at most c times the sum of the areas of the disks in $D \setminus D_\kappa$. As the increase in cost incurred by Cover(X, Y, κ) in turning the κ'-cover to a κ-cover is at most c times the sum of the areas of ths disks in D_κ, the theorem follows.

We now describe how D_k is computed, and then establish that it has the above two properties. For each $x' \in X'$, let largest(x') be the largest disk from D that contains x'. Since x' is $\kappa(x')$-covered by D, we note that the radius of largest(x') is at least $||x' - y^\kappa(x')||$. Let

$$D'_\kappa = \{\text{largest}(x') \mid x' \in X'\}.$$

Sort the disks in D'_κ by decreasing (non-increasing) radii. Let $B \leftarrow \emptyset$ initially. For each disk $d \in D'_\kappa$ in the sorted order, performing the following operation: add d to B if d does not intersect any disk already in B.

Let D_κ be the set B at the end of this computation. Since no two disks in D_κ intersect, and D κ-covers any point in X, it follows that $D \setminus D_\kappa$ κ'-covers any point in X. This establishes Property 2 of D_κ.

To show that the cost increase incurred by Cover(X, Y, κ) in going from the κ'-cover to the κ-cover is at most c times the sum of the α-th powers of the radii of the disks in D_κ (Property 1), it suffices, by Lemma 1, to show that $\sum_y \rho_y^\alpha$ is at most c' times the sum of the α-th powers of the radii of the disks in D_κ. Here, $c' > 0$ is an absolute constant.

For this, consider L_κ, the set of disks obtained by increasing the radius of each disk in D_κ by a factor of 3. By construction of D_κ, it can be seen that for any $x' \in X'$ there is a disk in L_κ with radius at least $||x' - y^\kappa(x')||$ containing x'. In other words, L_κ is a feasible solution to the optimization problem that the procedure OuterCover(X', Y, κ) solves. Since the procedure OuterCover(X', Y, κ) returns an $O(1)$ approximation to the optimal solution, we conclude that $\sum_y \rho_y^\alpha$ is within a multiplicative constant of the sum of the α-th powers of the radii of the disks in L_κ, and hence D_κ.

This establishes Property 1, and completes the proof of the theorem. □

We conclude with a statement of the main result of this note. Recall that the the cost of a cover here refers to the sum of the α-th power of the radii, for any constant $\alpha \geq 1$.

Theorem 2. *Given point sets X and Y in the plane and $\kappa : X \to \{0, 1, 2, \ldots, |Y|\}$, the algorithm Cover$(X, Y, \kappa)$ runs in polynomial time and computes a κ-cover of X with cost at most $O(1)$ times that of the optimal κ-cover.*