The Gale-Shapley Algorithm

During the course of the algorithm, certain pairs \((m, w) \in M \times W\) become "engaged". The set of engaged pairs forms a matching. We will say that an individual is engaged to mean that he/she is part of a pair that is engaged. An individual is said to be free if he/she is not engaged.
Initially, all $m \in M$ and $w \in W$ are free. While there is a man $m$ who is free and hasn't proposed to every woman

Choose such a man $m$

let $w \leftarrow$ highest-ranked woman in $m$'s list to whom $m$ has not yet proposed.

$m$ makes a proposal to $w$:

gf $w$ is free then $(m, w)$ become engaged.

Else let $m'$ be woman to whom $w$ is currently engaged.

gft $w$ prefers $m'$ to $m$ then $m$ remains free.

Else $w$ prefers $m$ to $m'$.

$(m, w)$ become engaged.

$(m', w)$ become disengaged, and so $m'$ becomes free.

Endgf

Endif

Endwhile

Return the set of engaged pairs.
Analysis of Algorithm.

Observation 1: The G-S Algorithm terminates after at most \( n^2 \) iterations of the while loop.

Proof: Within each while loop, some man \( m \) makes a proposal to some woman \( w \). Observe that \( m \) has never made a proposal to \( w \) before. Why?

This means that a given man \( m \) proposes to a given woman \( w \) at most once. There are exactly \( n^2 \) man-woman pairs, so at most \( n^2 \) proposals are made, so there are at most \( n^2 \) iterations of while-loop.
The above observation hints that a bound on the number of steps that an implementation of the algorithm would take. Note that what is needed to implement the steps within a the while loop is basically a book-keeping mechanism. More on this later.

We now show that the less obvious facts that the G-S algorithm returns (a) a perfect matching, and (b) a stable matching.

Observation 2: Fix some woman $w$. $w$ remains engaged from the point at which she receives her first proposal. The sequence of partners to which she is engaged gets better and better (in her ordering).
Observation 3 Let m be any man. Suppose m is free just before the execution of the while statement. Then there is a woman to whom he has not yet proposed.

Proof: Suppose the conclusion is false and m has proposed to all women. Then by Observation 2, all women are currently engaged. Since the set of engaged pairs form a matching, this means all that n men, thus all men, are currently engaged. So m is currently also engaged, and this contradicts the assumption that he is currently free.
Observation 4: The set \( S \) returned at termination forms a perfect matching.

Proof: The terminating condition of the while loop means that at termination there is no man who is free and has not proposed to a woman. Due to Observation 3, this means that there is no man who is free (at termination). So all men are engaged at termination, and the set of engaged pairs forms a perfect matching.
Observation 5: The set $S$ returned by an execution of the G-S algorithm is a stable matching.

Proof: We know that $S$ is a perfect matching. We will show that there is no instability with respect to $S$.

Let $(m, w') \in M \times W$ be any pair not in $S$. So $(m, w) \in S$ for some $w \neq w'$, and $(m', w') \in S$ for some $m' \neq m$.

\[ \begin{array}{c}
  m \\ \\
  m' \\
\end{array} \quad \begin{array}{c}
  w' \\ \\
  w \\
\end{array} \]

If $m$ did not propose to $w'$, then we can conclude that $m$ prefers $w$ to $w'$, because $m$'s proposals are ordered by his preference. So there is no danger of $(m, w')$ being an instability.
If $m$ did propose to $w'$, then since $m$ is not currently engaged to $w'$, $w'$ rejected $m$ (either at the time of $m$'s proposal or by later breaking engagement with $m$) in favor of $m''$ to whom she was engaged.

Either $m' = m''$, or by observation 2 $w'$ prefers $m'$ to $m''$. In either case, we see that $w'$ prefers $m'$ to $m$, so $(m, w')$ can't be an instability.

We conclude that $S$ is stable. ❅