

**22S:105**  
**Statistical Methods and Computing**

**Confidence Intervals**

Lecture 12  
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**Point estimation**

If we had to guess a single number for the population mean  $\mu$ , our best "educated guess" is  $\bar{x}$ , the sample mean.

$\bar{x}$  is our *point estimate* of  $\mu$ .

How great is the uncertainty in this estimate?

**Confidence in estimation**

Example: Studying the quantitative skills of young Americans of working age

We might use the quantitative scores from the national Assessment of Educational Progress (NAEP) Young Adult Literacy Assessment Survey

- possible scores from 0 to 500
- in a recent year, 840 men aged 21 to 25 years were in NAEP sample
  - can be considered a simple random sample from the population of 9.5 million young men in this age range
- mean quantitative score:  $\bar{x} = 272$

What can we conclude about the population mean score  $\mu$  of all 9.5 million young men?

**Interval estimation: the prelude**

Recall essentials about sampling distribution of  $\bar{x}$ :

- The mean  $\bar{x}$  of 840 scores has a distribution that is close to normal (by the Central Limit Theorem)
- The mean of this normal sampling distribution is the same as the unknown mean  $\mu$  of the entire population
- The standard deviation of  $\bar{x}$  for a simple random sample of 840 men is  $\frac{\sigma}{\sqrt{840}}$ 
  - where  $\sigma$  is the standard deviation of individual NAEP scores among all young men

### If we knew $\sigma$ ...

Imagine that we know that the true population standard deviation of quantitative scores among all young men is  $\sigma = 60$ .

Then the standard deviation of  $\bar{x}$  is

$$\frac{\sigma}{\sqrt{n}} = \frac{60}{\sqrt{840}} = 2.1$$

Imagine also that we could choose *many* samples of size 840 and find the mean NAEP quantitative score from each one.

If we collect all these different  $\bar{x}$ s and display their distribution, we get the normal distribution with

- mean equal to the unknown  $\mu$
- standard deviation 2.1

### 95% confidence

Our sample of 840 young men gave  $\bar{x} = 272$ .

We say that we are *95% confident* that the unknown mean NAEP quantitative score for all young men lies between

$$\bar{x} - 4.2 = 272 - 4.2 = 267.8$$

and

$$\bar{x} + 4.2 = 272 + 4.2 = 276.2$$

Every sample would give slightly different values for this interval.

Why are we so confident that  $\mu$  lies in the interval we happened to get?

### Statistical confidence

- The 68-95-99.7 rule says that in 95% of all samples, the mean score  $\bar{x}$  for the sample will be within two standard deviations of the population mean score  $\mu$ .
  - So the  $\bar{x}$  will be within 4.2 points of  $\mu$  in 95% of samples of 840 NAEP scores
- But if  $\bar{x}$  is within 4.2 points of the unknown  $\mu$ , then  $\mu$  also has to be within 4.2 points of the observed  $\bar{x}$ !
  - This will happen in 95% of all samples.
- That is, in 95% of all possible samples of size 840 from this population
  - the unknown  $\mu$  lies between  $\bar{x} - 4.2$  and  $\bar{x} + 4.2$

There are only two things that could have happened with our particular sample:

- We got a sample such that the true  $\mu$  does lie in our resulting interval. That is,  $\mu$  really is between 267.8 and 276.2.
- We were unlucky, and our simple random sample was one of the 5% of all possible samples where  $\bar{x}$  is not within 4.2 points of the true  $\mu$ .

We cannot know for sure which thing happened with our particular sample.

Saying “We are 95% confident that the unknown  $\mu$  lies in the interval (267.8, 276.2)” means

- “We got these numbers by a method that gives correct results 95% of the time.”

### What a 95% confidence interval does not mean

Saying “We are 95% confident that the unknown  $\mu$  lies in the interval (267.8, 276.2)” doesn't *not* mean

- $\mu$  is a random variable that has a value within the interval 95% of the time
- 95% of the population values lie in the interval

What if we wanted to be *more* confident that our interval contained  $\mu$ ?

We would use a *confidence level* other than 95%.

Example: we will compute a 99% confidence interval for the mean of NAEP quantitative scores in young men

We need the values for a standard normal distribution that cut off the top 0.005 and the bottom 0.005 of values.

- Table A.1 gives several possibilities (due to rounding).
- The most accurate choice is 2.58 for the upper cutoff.

So a 99% confidence interval for  $\mu$  would be

$$(\bar{x} - 2.58(2.1), \bar{x} + 2.58(2.1))$$

If we didn't need to be all that confident, how would we compute an 80% confidence interval for  $\mu$ ?

## Two-sided confidence intervals for a population mean

- Draw a simple random sample of size  $n$  from a population having
  - unknown mean  $\mu$
  - known standard deviation  $\sigma$
- A level  $C$  confidence interval for  $\mu$  is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

where  $z^*$  is the value that cuts off the upper  $\frac{1}{2}$  of  $(1 - C)$  of the area of a standard normal distribution

$z^*$  is called the *critical value*

## Critical values for the most commonly used confidence levels

Confidence level	Tail area	$z^*$
90%	0.05	1.645
95%	0.025	1.960
99%	0.005	2.576

## What affects the width of a confidence interval

The width of a confidence interval gets *smaller* if

- The confidence coefficient gets smaller (equivalently, if the level of confidence gets smaller)
- $\sigma$  gets smaller
- $n$  gets larger

## One-sided confidence intervals

What if we only need to be confident that  $\mu$  is below some upper bound (or above some lower bound), but we don't care how far it might be in the opposite direction?

Example: We are concerned that  $\mu$  for the NAEP scores might be very low, so we want to find a lower bound. That is, we want to find a value  $m$  such that we are 95% confident that  $\mu \geq m$ .

Begin by drawing the picture!

Now we will use Table A to find the value that cuts off the lower 5% of the area under a standard normal curve.

This is -1.645.

Therefore, we are 95% confident that  $\mu \geq \bar{x} - 1.645 \frac{\sigma}{\sqrt{n}}$ .

In other words, our one-sided confidence interval for  $\mu$  is

$$\begin{aligned}\mu &\geq \bar{x} - 1.645 \frac{\sigma}{\sqrt{n}} \\ &\geq 272 - 1.645 (2.1) \\ &\geq 268.55\end{aligned}$$