

STAT:2010/4200
Statistical Methods and Computing

Confidence Intervals

Lecture 12
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Point estimation

If we had to guess a single number for the population mean μ , our best "educated guess" is \bar{x} , the sample mean.

\bar{x} is our *point estimate* of μ .

How great is the uncertainty in this estimate?

Confidence in estimation

Example: Studying the quantitative skills of young Americans of working age

We might use the quantitative scores from the national Assessment of Educational Progress (NAEP) Young Adult Literacy Assessment Survey

- possible scores from 0 to 500
- in a recent year, 840 men aged 21 to 25 years were in NAEP sample
 - can be considered a simple random sample from the population of 9.5 million young men in this age range
- mean quantitative score: $\bar{x} = 272$

What can we conclude about the population mean score μ of all 9.5 million young men?

Interval estimation: the prelude

Recall essentials about sampling distribution of \bar{x} :

- The mean \bar{x} of 840 scores has a distribution that is close to normal (by the Central Limit Theorem)
- The mean of this normal sampling distribution is the same as the unknown mean μ of the entire population
- The standard deviation of \bar{x} for a simple random sample of 840 men is $\frac{\sigma}{\sqrt{840}}$
 - where σ is the standard deviation of individual NAEP scores among all young men

If we knew σ ...

Imagine that we know that the true population standard deviation of quantitative scores among all young men is $\sigma = 60$.

Then the standard deviation of \bar{x} is

$$\frac{\sigma}{\sqrt{n}} = \frac{60}{\sqrt{840}} = 2.1$$

Imagine also that we could choose *many* samples of size 840 and find the mean NAEP quantitative score from each one.

If we collect all these different \bar{x} s and display their distribution, we get the normal distribution with

- mean equal to the unknown μ
- standard deviation 2.1

95% confidence

Our sample of 840 young men gave $\bar{x} = 272$.

We say that we are *95% confident* that the unknown mean NAEP quantitative score for all young men lies between

$$\bar{x} - 4.2 = 272 - 4.2 = 267.8$$

and

$$\bar{x} + 4.2 = 272 + 4.2 = 276.2$$

Every sample would give slightly different values for this interval.

Why are we so confident that μ lies in the interval we happened to get?

Statistical confidence

- The 68-95-99.7 rule says that in 95% of all samples, the mean score \bar{x} for the sample will be within two standard deviations of the population mean score μ .
 - So the \bar{x} will be within 4.2 points of μ in 95% of samples of 840 NAEP scores
- But if \bar{x} is within 4.2 points of the unknown μ , then μ also has to be within 4.2 points of the observed \bar{x} !
 - This will happen in 95% of all samples.
- That is, in 95% of all possible samples of size 840 from this population
 - the unknown μ lies between $\bar{x} - 4.2$ and $\bar{x} + 4.2$

There are only two things that could have happened with our particular sample:

- We got a sample such that the true μ does lie in our resulting interval. That is, μ really is between 267.8 and 276.2.
- We were unlucky, and our simple random sample was one of the 5% of all possible samples where \bar{x} is not within 4.2 points of the true μ .

We cannot know for sure which thing happened with our particular sample.

Saying “We are 95% confident that the unknown μ lies in the interval (267.8, 276.2)” means

- “We got these numbers by a method that gives correct results 95% of the time.”

What a 95% confidence interval does not mean

Saying “We are 95% confident that the unknown μ lies in the interval (267.8, 276.2)” doesn't *not* mean

- μ is a random variable that has a value within the interval 95% of the time
- 95% of the population values lie in the interval

What if we wanted to be *more* confident that our interval contained μ ?

We would use a *confidence level* other than 95%.

Example: we will compute a 99% confidence interval for the mean of NAEP quantitative scores in young men

We need the values for a standard normal distribution that cut off the top 0.005 and the bottom 0.005 of values.

- Table A.1 gives several possibilities (due to rounding).
- The most accurate choice is 2.58 for the upper cutoff.

So a 99% confidence interval for μ would be

$$(\bar{x} - 2.58(2.1), \bar{x} + 2.58(2.1))$$

If we didn't need to be all that confident, how would we compute an 80% confidence interval for μ ?

Two-sided confidence intervals for a population mean

- Draw a simple random sample of size n from a population having
 - unknown mean μ
 - known standard deviation σ
- A level C confidence interval for μ is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

where z^* is the value that cuts off the upper $\frac{1}{2}$ of $(1 - C)$ of the area of a standard normal distribution

z^* is called the *critical value*

Critical values for the most commonly used confidence levels

Confidence level	Tail area	z^*
90%	0.05	1.645
95%	0.025	1.960
99%	0.005	2.576

What affects the width of a confidence interval

The width of a confidence interval gets *smaller* if

- The confidence coefficient gets smaller (equivalently, if the level of confidence gets smaller)
- σ gets smaller
- n gets larger

One-sided confidence intervals

What if we only need to be confident that μ is below some upper bound (or above some lower bound), but we don't care how far it might be in the opposite direction?

Example: We are concerned that μ for the NAEP scores might be very low, so we want to find a lower bound. That is, we want to find a value m such that we are 95% confident that $\mu \geq m$.

Begin by drawing the picture!

Now we will use Table A to find the value that cuts off the lower 5% of the area under a standard normal curve.

This is -1.645.

Therefore, we are 95% confident that $\mu \geq \bar{x} - 1.645 \frac{\sigma}{\sqrt{n}}$.

In other words, our one-sided confidence interval for μ is

$$\begin{aligned}\mu &\geq \bar{x} - 1.645 \frac{\sigma}{\sqrt{n}} \\ &\geq 272 - 1.645 (2.1) \\ &\geq 268.55\end{aligned}$$