1. Let $X$ be a discrete random variable with its probability function given by $P(X = x) = p q^x$, for any non-negative integer $x$ and where $0 < p < 1$ and $q = 1 - p$.

(a) Show that the mgf of $X$ equals $M(t) = \frac{1}{1-e^t q}$.

$$M(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{p q^x}{x!} = \frac{p}{1-e^t q}$$

(b) Find the mean $E(X)$.

$$E(X) = M'(t) = \frac{p q e^t}{(1-e^t q)^2}$$

$$E(X) = \frac{p q}{(1-q)^2} = \frac{q}{p}$$
2. Let $X$ and $T$ be two random variables. $T$ has a Gamma distribution with parameters $\alpha$ and $\beta$. Given $T = t$, $X$ has a Poisson distribution with mean equal to $t$.

(a) Compute the mean of $X$.

$$E(X) = E(E(X \mid T)) = E(T) = \alpha \beta$$

(b) Find $P(X = 0 \mid T = 5)$.

$$= \sum_{0}^{5} \frac{e^{-5}}{0!} = e^{-5}$$
(c) Find $P(X = 0)$.

$$P(X=0) = E\left( P(X=0 \mid \mathbf{X}) \right)$$

$$= E(e^{-\mathbf{X}})$$

$$= \frac{1}{\prod (1 - \beta e^{\mathbf{X}})^{\mathbf{x}}}$$

$$= \frac{1}{(1+\beta)^{\mathbf{x}}}.$$
3. Let $X \sim \text{Bin}(n = 40, p = 0.5)$.

(a) Use Central Limit Theorem to compute $P(X \geq 25)$ approximately. (Remember to use continuity correction.)

\[
P(X \geq 25) = P\left( X \geq \frac{24.5}{\sqrt{40 \times 0.5 \times 0.5}} \right)
\]

\[
\approx P\left( Z \geq 1.42 \right) = 1 - 0.9222 = 0.0778
\]

(b) Find $c$ such that $P(X \leq c) = 0.05$ approximately.

\[
0.05 = P\left( \frac{X - 40 \times 0.5}{\sqrt{40 \times 0.5 \times 0.5}} \leq \frac{c+\frac{1}{2} - 40 \times 0.5}{\sqrt{40 \times 0.5 \times 0.5}} \right)
\]

\[
\Rightarrow \frac{c+\frac{1}{2} - 40 \times 0.5}{\sqrt{40 \times 0.5 \times 0.5}} = -1.645
\]

\[
\Rightarrow c = 24.7 \quad 14.298
\]
4. Let $X_1, X_2$ be iid $N(0,1)$.

(a) Find the distribution of $Y = X_1 - 2X_2$. (You have to specify all parameters of the distribution.)

\[
E(Y) = 0 - 2 \cdot 0 = 0
\]

\[
\text{Var}(Y) = \text{Var}(X_1) + 4 \cdot \text{Var}(X_2) = 5
\]

\[
Y \sim N(0, 5)
\]

(b) Find the constant $c$ such that $W = X_1 + cX_2$ is independent of $Y$.

\[
\text{Ind} \quad \Rightarrow \quad Y \perp W
\]

\[
0 = \text{Cov}(Y, W) = \text{Cov}(X_1 - 2X_2, X_1 + cX_2)
\]

\[
= \text{Cov}(X_1, X_1) - 2c \cdot \text{Cov}(X_2, X_2)
\]

\[
= 1 - 2c
\]

\[
\Rightarrow \quad c = \frac{1}{2}
\]
5. The OraQuick test is a fast method for detecting HIV. The test is positive with probability 0.09 if the subject (person) receiving the test has HIV. On the other hand, it has a false alarm rate of 0.004, i.e., the test is positive with probability 0.004 even if the subject does not have HIV. About 4% of the young adults in the US have HIV.

(a) Find the probability that a random young adult is tested positive with the OraQuick test.

\[
P(\text{positive}) = 0.04 \times 0.99 + 0.96 \times 0.004
\]
\[
= 0.0434
\]

(b) Find the probability a person actually has HIV given that the person is tested positive with the OraQuick test.

\[
P(\text{HIV | positive}) = \frac{P(\text{HIV} \& \text{positive})}{P(\text{positive})}
\]
\[
= \frac{0.04 \times 0.99}{0.04344}
\]
\[
= 0.9116
\]
6. Let $X_1, X_2, \ldots, X_{100}$ be iid Poisson distributed with the population mean equal to 1.

(a) Find $P(\max(X_1, X_2, \ldots, X_{100}) > 0)$.

$$
P(\max(X_1, \ldots, X_{100}) > 0) = 1 - P(\max(X_1, \ldots, X_{100}) \leq 0)
= 1 - \prod_{i=0}^{100} P(X_i = 0)
= 1 - (e^{-1})^{100}
= 1 - e^{-100}.
$$

(b) Find the probability $P(\sum_{i=1}^{100} X_i \geq 120)$ approximately.

$$
P(\sum_{i=1}^{100} X_i \geq 120) = P\left( \frac{\sum_{i=1}^{100} X_i - 100 \mu}{\sqrt{100}} \geq \frac{120 - 100 \cdot 1}{10} \right)
\approx P(2 \geq \frac{19.5}{10})
= 1 - 0.9744
= 0.0256.
$$
Let $X_1, X_2, X_3$ and $X_4$ are iid $N(\mu, \sigma^2)$.

(a) Find $k$ such that $k(X_3 - X_4)^2/\sigma^2$ is $\chi^2(1)$.

\[
X_3 - X_4 \sim N(0, \sigma^2)
\]

\[
\left( \frac{X_3 - X_4}{\sqrt{\sigma^2}} \right)^2 \sim \chi^2(1)
\]

\[
\frac{(X_3 - X_4)^2}{\sigma^2} \sim \chi^2(1) \quad \text{as} \quad k = \frac{1}{2}
\]

(b) Find the constant $c$ such that $\frac{(X_1 - X_2)^2}{|X_3 - X_4|}$ is a $t$-distribution. What is the d.f. of the $t$-distribution?

\[
X_1 - X_2 \sim N(0, \sigma^2)
\]

\[
\frac{X_1 - X_2}{\sqrt{X_3 - X_4}} = \sqrt{\frac{(X_3 - X_4)^2}{2}} \sim t(1)
\]

\[
\therefore \quad c = 1
\]
Let $X_1$ and $X_2$ have the joint pdf $f(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} 0 < x_1 < x_2 < \infty$ and zero elsewhere.

(a) Are $X_1$ and $X_2$ independent of each other?

No, because the support is not rectangular.

(b) Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 - X_1$.

$$0 < Y_1, Y_2 < \infty$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \left| \begin{array}{cc} x_1 = y_1 \\ x_2 = y_2 + x_1 = y_2 + y_1 \\ \end{array} \right|$$

$$= e^{-x_2} e^{-y_1} = e^{-y_2} e^{-y_2} = 1$$
(c) Compute the marginal pdf of $Y_1$.

\[
\begin{align*}
\int_{
\gamma_1
\begin{array}{c}
\gamma_1 \\
\end{array}}
\infty
\int f(y_1, y_2) \, dy_2
\end{align*}
\]

\[
= \int_0^\infty e^{-y_1} e^{-y_2} \, dy_2
\]

\[
= e^{-y_1} \left[ \int_0^\infty e^{-y_2} \, dy_2 \right]
\]

\[
= e^{-y_1} \left( -e^{-y_2} \right) \bigg|_0^\infty
\]

\[
= e^{-y_1}, \quad 0 < y_1 < \infty
\]

\[
\int_{\gamma_1} f(y_1) \, dy_1 = 0 \quad \text{for } y_1 < 0.
\]