THICKNESS OF KNOTS

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In this paper we study physical knots; that is, knots tied (as closed loops) in real pieces of rope, which have diameter. Intuitively, for a given diameter, one needs a certain minimum length of rope in order to tie a (non-trivial) knot, and (more vaguely), the more complicated the knot you want to tie, the more rope you need. To be specific, we can ask:

Question. Can you tie a knot in a one-foot length of one-inch rope?[1]

Experiment suggests that the answer is no, but that this is not far off the critical length; both G. Buck [B] (using rope) and A. Stasiak [S] (using computer simulation) have found that the minimum sufficient length for one-inch rope is approximately 16 inches. We show here (Corollary 3) that the length must at least be greater than 2.5π.

We need a mathematical model of a physical knot; our model of a knot in a rope of given diameter is a smooth curve having an embedded tubular neighborhood of that diameter. (See §1.) This is surely not the only possible model, but it seems to be a reasonable one. We also define the thickness of such a knot to be the ratio of its radius to its length. Then the above question becomes: does there exist a non-trivial knot of thickness at least $1/24$? (One-inch diameter = $1/2$ inch radius.) In §2 we relate thickness to curvature, and hence to bridge number, and show that a non-trivial knot must have thickness at most $1/4\pi$.

In §3 we relate thickness to the number of segments in a polygonal representative of the knot; this improves the bound for a non-trivial knot to $1/5\pi$, and also shows that there are only finitely many knots whose thickness is greater than a given positive number.

In §4 we also relate thickness to the self-distance [K] and distortion [G] of a knot.

The result on curvature was announced in [L, Si], and a weaker bound on the number of edges in [Si].

§1. Definitions and notation.

Throughout this paper, a *smooth knot* will mean a C^2 submanifold of $\mathbb{R}^3$ homeomorphic to S^1. Let K be a smooth knot. Then K has a C^2 parametrization by arc-length, $p\colon \mathbb{R} \to K$, where p has period $L = L(K)$, the length of K. We let $\mathbf{T}(s) = p'(s)$, the unit tangent vector to K at $p(s)$, and $\kappa(s) = ||\mathbf{T}'(s)||$, the curvature of K at $p(s)$. If the particular parametrization is not being emphasized, we may also use $\mathbf{T}_x$ to denote $\mathbf{T}(s)$, where $x = p(s)$. When $\kappa(s) \neq 0$, we also let $\mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s)$, the principal normal to K at $p(s)$. When $\kappa(s) = 0$, $\mathbf{N}(s)$ is undefined.

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[1]The question of thickness versus knot complexity was put, in this simple form, to one of the authors by L. Siebenmann in January 1985.

We are going to define the *injectivity radius* $R = R(K)$ of K, which is supposed to be the mathematical analog of the radius of the thickest piece of rope that can have K as its centerline. Intuitively, the content of the definition is this. For some radius $r > 0$, construct at each point x of K a standard disk of radius r centered at x in the plane normal to K at x. For small enough r, these disks are pairwise disjoint and form a solid tube around K. Let $R(K)$ be the supremum of such "good" radii r. Formally, we consider the normal bundle of the embedding of K in $\mathbb{R}^3$, whose total space is

$$E = \{ (p(s), v) \in K \times \mathbb{R}^3 : \mathbf{T}(s) \cdot v = 0 \},$$

and the exponential map $\exp\colon E \to \mathbb{R}^3$, which is defined by $\exp(x, v) = x + v$.
For $r > 0$, we let

$$\bar{E}_r = \{ (x, v) \in E : ||v|| \le r \}$$

be the associated closed disk bundle of radius r. By the tubular neighborhood theorem, the restriction of exp to $\bar{E}_r$ is injective for sufficiently small r, so we may define

$$R(K) = \sup \{ r > 0 : \exp \text{ is injective on } \bar{E}_r \}.$$

We define the *thickness* of K to be $\tau(K) = R(K)/L(K)$, and the thickness of a tame knot type $\mathcal{K}$ to be $\tau(\mathcal{K}) = \sup\, \tau(K)$, the supremum being taken over all smooth representatives of $\mathcal{K}$.

Remark 1. The tubular neighborhood theorem guarantees that, for small r, exp is not only injective but also a C^1 embedding on $\bar{E}_r$. Thus an alternative definition of injectivity radius is

$$R'(K) = \sup \{ r > 0 : \exp \text{ is a } \mathrm{C}^1 \text{ embedding on } \bar{E}_r \}.$$

It is evident that $R'(K) \le R(K)$; it will follow from Lemma 1 below that in fact $R'(K) = R(K)$. ∘

Remark 2. In [N], Nabutovsky defines a notion of thickness for knots of any dimension and codimension. In the classical dimension, it is not hard to see that his $r(K)$ is equal to our $R(K)$. Nabutovsky deals mostly with hypersurfaces, but in §4.2 makes some remarks about the classical case. In particular, his question D is the one we address here.

In [DEJ], a notion of thickness for knots in $\mathbb{R}^3$ is also defined, which, however, differs from ours. Denoting the thickness defined there by $\tau'(K)$, we easily have that $\tau(K) \le \tau'(K)$. On the other hand, $\tau'(K)$ is shown in [DEJ] to be continuous with respect to the C^1 topology, which $\tau(K)$ is not. (This follows from our Theorem 1; one can introduce into any knot a point of high curvature by a C^1-small deformation.) ∘

We further let

$$E_r = \{ (x, v) \in E : ||v|| < r \} \quad \text{and}$$
$$S_r = \{ (x, v) \in E : ||v|| = r \}$$

be the open disk and circle bundles of radius r in E. The fibers of the bundles E, $\bar{E}_r$, E_r, and S_r over $x \in K$ will be denoted by $E(x)$, $\bar{E}_r(x)$, $E_r(x)$, and $S_r(x)$. We let

$$P(x) = \exp(E(x)) \quad \text{and}$$
$$D_r(x) = \exp(\bar{E}_r(x))$$

be the plane and (closed) disk of radius r normal to K at x.

When we need to take a derivative, we use the least sophisticated version that the situation will allow. Since this varies from place to place, there is some potential for confusion, so we spell out our notation here. If $f\colon M \to N$ is a differentiable map between manifolds of class at least C^1, then $Df(x)$ is a linear map $T_xM \to T_{f(x)}N$ between tangent spaces. It will always happen that either $M = \mathbb{R}$ or $N = \mathbb{R}^3$. If $N = \mathbb{R}^3$, we identify $T_{f(x)}N$ with $\mathbb{R}^3$ (in the canonical way), so that $Df(x)$ becomes a linear map $T_xM \to \mathbb{R}^3$. If $M = \mathbb{R}$, we likewise identify T_xM with $\mathbb{R}$, and set $f'(x) = Df(x)(1)$, an element of $T_{f(x)}N$. If both $M = \mathbb{R}$ and $N = \mathbb{R}^3$, then $f'(x)$ is a vector in $\mathbb{R}^3$, and we are operating at the level of a multi-variable calculus course.

§2. Thickness and curvature.

We associate two more numbers R_1 and R_2 to a smooth knot K. First, let $R_1 = R_1(K) = 1/\max \kappa(s)$, the minimum radius of curvature. Second, let $C(K) \subset K \times K$ be the set of all pairs $(x_1, x_2) = (p(s_1), p(s_2))$ with $x_1 \neq x_2$ and $(x_2 - x_1) \cdot T(s_1) = 0 = (x_2 - x_1) \cdot T(s_2)$. Note that $C(K)$ is disjoint from some neighborhood of the diagonal in $K \times K$, and hence is closed. Moreover, $C(K)$ is non-empty because it contains any pair (x_1, x_2) for which $||x_2 - x_1||$ is a maximum. Thus we may set

$$R_2 = R_2(K) = \frac{1}{2} \min \{\, ||x_2 - x_1|| : (x_1, x_2) \in C(K) \,\}.$$

We can now state the main result of this section.

Theorem 1. *For any smooth knot K, $R(K) = \min \{\, R_1(K), R_2(K) \,\}$.*

From the part of Theorem 1 which says $R \leq R_1$, which is equivalent to $\kappa(s) \leq 1/R$ for all s, we deduce two corollaries. Both of these will be improved in the next section. Also in Theorem 4 (§4), we give an alternative formulation of Theorem 1, in terms of the self-distance of a knot.

Corollary 1. *If $\mathcal{K}$ is a tame knot type, then $\tau(\mathcal{K}) \leq 1/(2\pi\, \mathrm{br}(\mathcal{K}))$, where $\mathrm{br}(\mathcal{K})$ is the bridge number of $\mathcal{K}$.*

Proof. By the preceding remark, any smooth representative K of $\mathcal{K}$ has total curvature at most $1/\tau(K)$. The result follows using Corollary 3.2 of [M]. □

Corollary 2. *If $\mathcal{K}$ is a non-trivial knot type, then $\tau(\mathcal{K}) \leq 1/4\pi$.* □

We need some lemmas for the proof of Theorem 1.

Lemma 1. *Let (x_0, v_0) be a point of the normal bundle E of the smooth knot K, with $x_0 = p(s_0)$. Then (x_0, v_0) is a critical point of the exponential map if and only if the curvature $\kappa(s_0)$ is non-zero and $v_0 \cdot \mathbf{N}(s_0) = 1/\kappa(s_0)$. Further, at a critical point, $\exp$ is not locally injective, so $R'(K) = R(K)$; that is if r is such that the exponential map is injective on $\bar{E}_r$ then it is a C^1 embedding on $\bar{E}_r$.*

This type of result apparently is familiar to geometers in the context of hypersurfaces; see [DC] §10.4 and [T] Chapter 16. If, as is common in elementary geometry treatments of the local structure of curves, one assumes nonvanishing curvature (so the Frenet frame exists) and class C^3 (so the Frenet frame is C^1) then one can give a simpler proof than the following. For in this case, the principal normal and binormal give a trivialization of the normal bundle, which together with the parametrization of K defines local coordinates on E. In these coordinates, the derivative of exp can be computed using the Frenet formulas.

Proof of Lemma 1. In the tangent space $T_{(x_0,v_0)}E$ to E at (x_0, v_0), we have the subspace $T_{(x_0,v_0)}E(x_0)$ tangent to the fiber. Since exp maps the fiber $E(x_0)$ onto the normal plane $P(x_0)$ to the knot by an affine isomorphism, $D\exp(x_0, v_0)$ maps the tangent space to the fiber isomorphically onto the subspace of $\mathbb{R}^3$ orthogonal to $\mathbf{T}(s_0)$. Hence (x_0, v_0) is a critical point of exp iff for some (hence any) element ξ of $T_{(x_0,v_0)}E$ which is not in $T_{(x_0,v_0)}E(x_0)$, we have $D\exp(x_0, v_0)(\xi) \cdot \mathbf{T}(s_0) = 0$. Such a vector can be obtained as the tangent vector to a section of E. Let $v\colon \mathbb{R} \to \mathbb{R}^3$ be a C^1 function such that $v(s) \cdot \mathbf{T}(s) = 0$ for all s, and $v(s_0) = v_0$. (For instance, set $v(s) = v_0 - (v_0.\mathbf{T}(s))\mathbf{T}(s)$.) Then the curve $\gamma(s) = (p(s), v(s))$ in E is such that $\gamma'(s_0) \in T_{(x_0,v_0)}E$ is not tangent to the fiber. Now $\exp(\gamma(s)) = p(s) + v(s)$, so $D\exp(x_0, v_0)(\gamma'(s_0)) = (\exp\circ\gamma)'(s_0) = \mathbf{T}(s_0) + v'(s_0)$. Since $v(s) \cdot \mathbf{T}(s) = 0$ for all s, we have $v'(s) \cdot \mathbf{T}(s) + v(s) \cdot \mathbf{T}'(s) = 0$, and so

$$D\exp(x_0, v_0)(\gamma'(s_0)) \cdot \mathbf{T}(s_0) = 1 - v(s_0) \cdot \mathbf{T}'(s_0). \tag{2.1}$$

If $\kappa(s_0) = 0$ then this is 1, and (x_0, v_0) is not a critical point. If $\kappa(s_0) \neq 0$ then $v(s_0) \cdot \mathbf{T}'(s_0) = \kappa(s_0)v(s_0) \cdot \mathbf{N}(s_0)$, and (x_0, v_0) is a critical point iff $v_0 \cdot \mathbf{N}(s_0) = 1/\kappa(s_0)$, as claimed.

To prove the second claim of the lemma, orient E so that exp is orientation-preserving at each point of the zero section. Then the equation (2.1) shows that exp is orientation-preserving at a point (x_0, v_0) if $v(s_0) \cdot \mathbf{T}'(s_0) < 1$, and orientation-reversing if $v(s_0) \cdot \mathbf{T}'(s_0) > 1$. Therefore, in any neighborhood of a critical point there are points at which exp is orientation-preserving and points at which it is orientation-reversing. But if exp is locally injective at (x_0, v_0) then it is a local homeomorphism there (by invariance of domain), and therefore either orientation-preserving or orientation-reversing over an entire neighborhood, since these notions can be defined homologically. Thus the exponential map will fail to be injective at whatever radius it fails to be a local diffeomorphism. □

Lemma 2. *Let (x_0, v_0) be a point of the normal bundle E of the smooth knot K which is not a critical point of the exponential map. Suppose $r = ||v_0|| > 0$. Then the image under $D\exp(x_0, v_0)$ of the tangent space to the torus S_r at (x_0, v_0) is the subspace of $\mathbb{R}^3$ orthogonal to v_0.*

Proof. It suffices to show that, for any C^1 curve $\gamma(t) = (x(t), v(t))$ in S_r with $\gamma(0) = (x_0, v_0)$, we have $D\exp(x_0, v_0)(\gamma'(0)) \cdot v_0 = 0$. Now $D\exp(x_0, v_0)(\gamma'(0)) = x'(0) + v'(0)$, and v_0 is orthogonal to each summand: $x'(0) \cdot v_0 = 0$ since $x'(0)$ is tangent to K at x_0, and $v'(0) \cdot v_0 = 0$ since $v(t)$ has constant length r. □

Variants of the next lemma may be found (at least implicitly) almost anywhere the tubular neighborhood theorem is proved; see, for instance, Lemma 19 of Chapter 9 of [Sp]. For the convenience of the reader, we give a proof of a version adequate for our needs.

Lemma 3. *Let X and Y be Hausdorff spaces and $f\colon X \to Y$ a local homeomorphism. Let A be a compact subset of X such that $f|A$ is injective. Then there is a neighborhood U of A such that $f|U$ is injective.*

Proof. Let S be the subset of $X \times X$ consisting of pairs (x, y) with $x \neq y$ and $f(x) = f(y)$. Since f is a local homeomorphism, there is a neighborhood of the diagonal disjoint from S, and hence S is closed. Since $f|A$ is injective, $A \times A$ is disjoint from S, and since A is compact, there is a neighborhood U of A with $U \times U$ disjoint from S. Then $f|U$ is injective. □

Proof of Theorem 1. If $x_0 = p(s_0)$ is a point with $\kappa(s_0) = 1/R_1$, then by Lemma 1, exp fails to be locally injective at its critical point $(x_0, R_1\mathbf{N}(s_0))$, so $R \le R_1$. Recall the subset $C(K)$ of $K \times K$ from the definition of R_2. If $(x_1, x_2) \in C(K)$ with $||x_2 - x_1|| = 2R_2$, then the midpoint of x_1 and x_2 is in $D_{R_2}(x_1) \cap D_{R_2}(x_2)$, so $R \le R_2$.

It remains to prove that $R \ge \min\{R_1, R_2\}$. If $R \ge R_1$ there is nothing to do, so suppose that $R < R_1$. Then exp is a local homeomorphism on $E_{R_1} \supset \bar{E}_R$ (having no critical points there), and is injective on E_R. If exp were injective on S_R, it would follow from Lemma 3 that exp was injective on some neighbourhood of $\bar{E}_R$, and therefore on $\bar{E}_r$ for some $r > R$; this is a contradiction. Thus exp is not injective on the torus S_R. If the immersed torus $\exp(S_R)$ had any points of transverse self-intersection, then (since transversality is stable) exp would fail to be injective on S_r for all r sufficiently close to R, which is a contradiction for $r < R$. Thus $\exp(S_R)$ has tangential self-intersection, that is there exist (x_1, v_1) and (x_2, v_2) in S_R with $x_1 \neq x_2$, $x_1 + v_1 = x_2 + v_2$, and

$$D\exp(x_1, v_1)(T_{(x_1,v_1)}S_R) = D\exp(x_2, v_2)(T_{(x_2,v_2)}S_R).$$

In view of Lemma 2, this implies that $v_1 = -v_2$, so $x_2 - x_1 = 2v_1 = -2v_2$ and (x_1, x_2) is in $C(K)$ with $||x_2 - x_1|| = 2R$. Therefore $R \ge R_2$, and the proof is complete. □

§3. Thickness and polygonal representatives.

Theorem 2. *Let K be a smooth knot of thickness τ, and let n be an integer with $n > 1/\pi\tau$. Then K is equivalent to a polygonal knot with n segments.*

Remark. In one case at least, this is the best possible; if K is a circle then $\tau = 1/2\pi$, and the condition on n is $n > 2$, which cannot be improved. In [O], O'Hara proves a similar result involving Kuiper's self-distance. To compare these results, let $\operatorname{seg}(\mathcal{K})$ be the minimum number of segments in a polygonal representative of a knot type $\mathcal{K}$. Then O'Hara's result is that, for any smooth representative K of length 1, $\operatorname{seg}(\mathcal{K}) \le \lfloor 1/\operatorname{sd}(K) \rfloor + 1$. In view of Theorem 4 in the next section, this implies that $\operatorname{seg}(\mathcal{K}) \le \lfloor 1/2R(K) \rfloor + 1$. On the other hand, Theorem 2 can be written in the form $\operatorname{seg}(\mathcal{K}) \le \lfloor 1/\pi\tau(K) \rfloor + 1$, which for length 1 is $\operatorname{seg}(\mathcal{K}) \le \lfloor 1/\pi R(K) \rfloor + 1$. Thus neither result is a consequence of the other (at least not in an obvious way). Finally, we note that Theorem 2 implies Corollary 1, because $2\operatorname{br}(\mathcal{K}) + 1 \le \operatorname{seg}(\mathcal{K})$. In fact, for a non-trivial knot type, $2\operatorname{br}(\mathcal{K}) + 2 \le \operatorname{seg}(\mathcal{K})$ (see Lemma 4 below) so that Theorem 2 is stronger than Corollary 1. ∘

We note some consequences of Theorem 2 before giving its proof.

Corollary 3. *If $\mathcal{K}$ is a non-trivial knot type, then $\tau(\mathcal{K}) \leq 1/5\pi$.*

Corollary 4. *Given $\tau > 0$, there are only finitely many knot types $\mathcal{K}$ with $\tau(\mathcal{K}) \geq \tau$.*

Both corollaries are immediate from Theorem 2 and the following lemma, whose proof is essentially the same as one given in [R] (Theorem 1), and described there as 'perhaps that of Kuiper'.

Lemma 4. *Let $\mathcal{K}$ be a knot type having a polygonal representative with $n \geq 4$ segments. Then $\mathcal{K}$ has crossing number at most $(n-1)(n-4)/2$. In particular, if $\mathcal{K}$ is non-trivial then $n \geq 6$. Also $2\,\mathrm{br}(\mathcal{K}) + 2 \leq \mathrm{seg}(\mathcal{K})$.*

Proof. Let $K \subseteq \mathbb{R}^3$ be a polygonal representative of $\mathcal{K}$ with n segments. We may assume that one of the segments is parallel to the third coordinate axis. Let $\nu\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection onto the first two coordinates. Then $\nu(K)$ is a union of $n-1$ line segments. After a small isotopy of K, we may assume that $\nu(K)$ is a regular projection of some representative of $\mathcal{K}$ (though not, of course, of K). In this regular projection, the number of crossings is at most the number of unordered pairs of non-adjacent segments, which is $(n-1)(n-4)/2$. The second statement follows since $n = 4, 5$ give $(n-1)(n-4)/2 = 0, 2$ respectively.

The argument for bridge number is similar to the preceeding paragraph. With $\nu(K)$ a union of $(n-1)$ line segments, rotate K about the projection axis until at least one of the nondegenerate vertices in the projection is neither a local maximum nor a local minimum. Use a slight tilt of the axis to obtain K with a regular projection in which now at least two

vertices are neither maxima nor minima. Then the number of local maxima for this projection is at most $(n-2)/2$.

□

Proof of Theorem 2. Let L be the length of K, and divide K into n arcs of length $\ell = L/n$. Also let $R = R(K)$. Then $\ell < \pi R$. Our aim is to show that the inscribed polygonal knot K' given by this subdivision lies inside the embedded tube $\exp(E_R)$, and is transverse to the fibers. (At a vertex, this means that the two adjacent segments make non-zero angles with the fiber on opposite sides.) Then, by a compactness argument, K' lies in the interior of an embedded closed tube $\exp(\bar{E}_r)$ for some $r < R$, and so there is a homeomorphism of $\mathbb{R}^3$ taking K to K' which is the identity outside $\exp(E_r)$.

Let A be one of the arcs of K; we may assume that our parametrization p is chosen so that $A = p([0, \ell])$. Let $a = p(0)$ and $b = p(\ell)$ be the endpoints of A. We shall denote the line through a and b by ab, and the chord of K with endpoints a and b by $\widehat{ab}$. For $s_1, s_2 \in [0, \ell]$, let $\theta(s_1, s_2) = \cos^{-1}(\mathbf{T}(s_1) \cdot \mathbf{T}(s_2))$ be the angle between $\mathbf{T}(s_1)$ and $\mathbf{T}(s_2)$. We show that

$$\theta(s_1, s_2) \leq |s_2 - s_1|/R. \tag{3.1}$$

We may assume that $s_1 \leq s_2$. Observe that $\int_{s_1}^{s_2} \kappa(s)\,ds$ is the length of the tangent spherical image of the sub-arc $p([s_1, s_2])$ of A, while $\theta(s_1, s_2)$ is the angle subtended at the center of the unit sphere by the endpoints of this spherical image. Hence, using Theorem 1,

$$\begin{aligned}\theta(s_1, s_2) &\le \int_{s_1}^{s_2} \kappa(s)\, ds \\ &\le \int_{s_1}^{s_2} 1/R\, ds \\ &= (s_2 - s_1)/R,\end{aligned}$$

and (3.1) is proved.

From (3.1), we have (since $|s_2 - s_1|/R \le \ell/R < \pi$) that $\mathbf{T}(s_1) \cdot \mathbf{T}(s_2) \ge \cos(s_2 - s_1)/R$, and so

$$\begin{aligned}\mathbf{T}(s_1) \cdot (b - a) &= \int_0^{\ell} \mathbf{T}(s_1) \cdot \mathbf{T}(s)\, ds \\ &\ge \int_0^{\ell} \cos(s - s_1)/R\, ds \\ &= R\big(\sin s_1/R + \sin(\ell - s_1)/R\big).\end{aligned} \tag{3.2}$$

Since s_1/R and $(\ell - s_1)/R$ are both in the interval $[0, \pi)$ and are not both zero, we find that

$$\mathbf{T}(s_1) \cdot (b - a) > 0. \tag{3.3}$$

For future use (proof of Theorem 3), we note that (3.2) remains valid for $\ell = \pi R$.

Next, for $s \in [0, \ell]$, let $q(s)$ be the orthogonal projection of $p(s)$ on the line ab. The image of $[0, \ell]$ under q contains the chord $\widehat{ab}$ (being connected and containing a and b). In fact, it follows from (3.3) that q maps $[0, \ell]$ onto $\widehat{ab}$ by a C^1 diffeomorphism, but we make no use of this. Define d_A to be the maximum value of $||p(s) - q(s)||$. We shall show that $d_A < R$. The maximum occurs at (at least one) interior point s_0 of $[0, \ell]$. We may assume (by reversing the parametrization if necessary) that $s_0 \le \ell/2$. At s_0 we have $(p(s_0) - q(s_0)) \cdot (\mathbf{T}(s_0) - q'(s_0)) = 0$; since $q'(s_0)$ is parallel to ab, this gives $(p(s_0) - q(s_0)) \cdot \mathbf{T}(s_0) = 0$. (That is, the maximum occurs along a common perpendicular of A and $\widehat{ab}$.) Now let ν be the orthogonal projection of $\mathbb{R}^3$ on the normal plane $P(p(s_0))$ to K. The line segment joining $p(s_0)$ to $q(s_0)$ lies in this plane, and the line ab is mapped by ν either to the line in $P(p(s_0))$ orthogonal to this segment, or to the single point $q(s_0)$. In either case, $d_A = ||p(s_0) - q(s_0)|| \le ||p(s_0) - \nu(a)||$, which in turn is less than or equal to the length of the curve $\nu \circ p$ restricted to $[0, s_0]$. To compute this length, observe that $||(\nu \circ p)'(s)|| = ||\nu(\mathbf{T}(s))|| = \sin\theta(s, s_0)$. Since, for $0 \le s \le s_0$, $(s_0 - s)/R \le \ell/2R < \pi/2$, the inequality (3.1) gives $\sin\theta(s, s_0) \le \sin(s_0 - s)/R$. Therefore

$$\begin{aligned}d_A &\le \int_0^{s_0} \sin(s_0 - s)/R\, ds \\ &= R(1 - \cos s_0/R) \\ &< R.\end{aligned}$$

We next claim that every point of the chord $\widehat{ab}$ lies in the normal disk $D_{d_A}(p(s))$ at some point of A. (The point $p(s)$ is unique since d_A is less than the injectivity radius.) This is clear for the endpoints of $\widehat{ab}$, so let c be an interior point. Let $f(s) = ||p(s) - c||^2$. Then $f'(s) = 2(p(s) - c) \cdot \mathbf{T}(s)$, so (by (3.3)) $f'(0) < 0$ and $f'(\ell) > 0$. Hence the minimum of $f(s)$, and therefore of $||p(s) - c||$, occurs at some interior point s_1 of $[0, \ell]$, and $(p(s_1) - c) \cdot \mathbf{T}(s_1) = 0$. Thus c does lie in a normal plane $P(p(s_1))$. Further $c = q(s_2)$ for some s_2, and so $||p(s_1) - c|| \leq ||p(s_2) - q(s_2)|| \leq d_A$, and our claim is proved.

We have now shown that the chord $\widehat{ab}$ lies in the union of the normal disks of radius d_A to K at points of A, and thus in the image under exp of the part of E_R lying over A. Therefore the polygonal knot K' does lie in $\exp(E_R)$. Further, (3.3) shows that K' is transverse to the fibers (with the meaning explained above at the vertices), and we are done. □

§4. Thickness, distortion, and self-distance.

In this section, we relate the thickness $\tau(K)$ to two other measures of how "close" a knot gets to itself: the *distortion* of K [G] and the *self-distance* of K [K].

For any points $x, y \in K$, we can measure $||x - y||$, the straight-line distance between x and y in $\mathbb{R}^3$; and we can measure the minimum distance between x and y along the curve K, which we denote $\alpha(x, y)$.

For points x, y that are near each other along K, the ratio $\frac{\alpha(x,y)}{||x-y||}$ is close to 1 (a proof of this well-known fact is included in the proof below). Thus the function $K \times K \to \mathbb{R}^1$

given by $(x, y) \to \frac{\alpha(x,y)}{||x-y||}$ for $x \neq y$ and 1 for $x = y$ is continuous on $K \times K$ and, in particular, bounded. The *distortion* of K is the maximum over $K \times K$ of $\frac{\alpha(x,y)}{||x-y||}$, some finite number > 1.

As noted in the proof of Theorem 2, when $x \approx y$, the chord vector $(y - x)$ and the tangent vectors $\mathbf{T}_x$ and $\mathbf{T}_y$ are nearly parallel; in particular, $(y - x) \cdot \mathbf{T}_y \neq 0$. On the other hand, for each $x \in K$, there must be some point(s) y for which $(y - x) \cdot \mathbf{T}_y = 0$, for example choose y to make $||y - x||$ maximum.

Call a pair of distinct points $(x, y) \in K \times K$ *critical* if $(y - x) \cdot \mathbf{T}_y = 0$. (Note that having (x, y) critical does not imply that (y, x) is critical, as the chord $(y - x)$ need not be perpendicular to $\mathbf{T}_x$.) The *self-distance* of K is

$$\mathrm{sd}(K) = \min \{ \, ||y - x|| : (x, y) \text{ is critical} \, \}.$$

Note that the self-distance of K is, in general, strictly less than the minimum (used in §2 to define R_2) over the "doubly-critical" pairs comprising the set $C(K)$. Consider, for example, an ellipse of high eccentricity. Thus Theorem 4 below may be a bit surprising.

Theorem 3. *If K is a knot with thickness $\tau = \tau(K)$, then* $\mathrm{distortion}(K) \leq \frac{1}{4\tau}$.

Proof. We shall assume in this proof not only that K is parametrized by arclength, so the norm of the derivative $||p'(t)|| = 1$, but also that K has been normalized to have

total length $= 1$, so $\alpha(x, y) \leq \frac{1}{2}$ and $\tau(k) = R(K)$.

We divide $K \times K$ into two sets, $A = \{ \, (x, y) : \alpha(x, y) \leq \pi R \, \}$ and $B = \{ \, (x, y) : \alpha(x, y) \geq \pi R \, \}$.

First consider points $(x, y) \in A$. If we set $\ell = \alpha(x, y)$, then we may choose our parametrization so that $x = p(0)$ and $y = p(\ell)$. By (3.2), we have

$$\mathbf{T}(s) \cdot (x - y) \geq R\big(\sin s/R + \sin(\ell - s)/R\big)$$

for $0 \leq s \leq \ell$. Integrating with respect to s gives

$$\begin{aligned} ||x - y||^2 &\geq 2R^2(1 - \cos \ell/R) \\ &= 4R^2 \sin^2(\ell/2R), \end{aligned}$$

and hence

$$\begin{aligned} \frac{\alpha(x, y)}{||x - y||} &= \frac{\ell}{||x - y||} \\ &\leq \frac{\ell/2R}{\sin(\ell/2R)} \\ &\leq \frac{\pi}{2}. \end{aligned} \tag{4.1}$$

(The last inequality holds because for any angle θ with $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{\theta}{\sin(\theta)} \leq \frac{\pi}{2}$). Since $\tau \leq \frac{1}{2\pi}$, we have $\frac{\alpha(x,y)}{||x-y||} \leq \frac{1}{4\tau}$ for $(x, y) \in A$.

Now consider points $(x, y) \in B$. Choose (x_0, y_0) to minimize the distance $||x - y||$ over B. We consider two sub-cases: $\alpha(x_0, y_0) > \pi R$ and $\alpha(x_0, y_0) = \pi R$. If $\alpha(x_0, y_0) > \pi R$ then (x_0, y_0) must be a critical point of the function $||x - y||$; that is, (x_0, y_0) is in the "doubly-critical" set $C(K)$, so $||x_0 - y_0|| \geq 2R_2 \geq 2R$. If $\alpha(x_0, y_0) = \pi R$ then (x_0, y_0) is also in A, so by (4.1) $||x_0 - y_0|| \geq 2\alpha(x_0, y_0)/\pi = 2R$. Hence we have $||x - y|| \geq 2R = 2\tau$ for any $(x, y) \in B$. Since also $\alpha(x, y) \leq \frac{1}{2}$, we have $\frac{\alpha(x,y)}{||x-y||} \leq \frac{1}{4\tau}$. □

Theorem 4. *(cf. Theorem 1) For any smooth knot K,*

$$R(K) = \min\{\, R_1(K), \frac{1}{2}\,\mathrm{sd}(K)\,\}.$$

Remark. Another way to interpret this theorem is that when thickness is being controlled by (doubly-critical) self-distance, as opposed to curvature, then the two self-distance minima agree even if, in general, singly critical self-distance $\lneq$ doubly critical self-distance. ○

Proof of Theorem 4. Because of Theorem 1 and the evident inequality $\mathrm{sd}(K) \leq 2R_2(K)$, what needs to be proved is that $R(K) \leq \frac{1}{2}\,\mathrm{sd}(K) = r$, say. Suppose that the self-distance is realised at the pair (x, y) with the chord perpendicular to the tangent at y. If the chord is also perpendicular to the tangent at x then $r = R_2(K)$, so suppose this is not the case. Let m be the midpoint of x and y, and let z be the point of K closest to m. Since the distance from m to a variable point of K does not have a local minimum at x, we have $||z - m|| < r$, and in particular $z \neq y$. But now m lies in the normal disks of radius r to K at both y and z, so $R(K) \leq r$. □

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