Suppose you are trying to solve a system of linear equations such as
\[
\begin{align*}
2x + 3y &= 4 \\
-x - y &= 5 \\
x + 2y &= 1 \\
-3x + 4y &= 7
\end{align*}
\]

Even though there are more equations than unknowns, we might still hope there exist solutions. We can form the augmented coefficient matrix, and row-reduce it. In this case,
\[
A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 5 \\ 1 & 2 & 1 \\ -3 & 4 & 7 \end{bmatrix} \implies \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The third row of \(\text{rref}(A)\) says "zero equals 1", which tells us that the system has no solution.

Let's look at the problem stated in terms of vectors. We seek coefficients \(\{x, y\}\) such that
\[
x \begin{bmatrix} 2 \\ 1 \\ 1 \\ -3 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 7 \end{bmatrix}.
\]

The desired output vector \([4, 5, 1, 7]^T\) does not lie in the span of the two vectors on the left side; that is the same thing as saying that there are no values for \(v\) and \(y\) to solve the system.

If we cannot find a linear combination of \([2, 1, 1, -3]^T\) and \([3, -1, 2, 4]^T\) that exactly equals the one on the right, we can try to find a linear combination that is as close as possible to the one on the right.

We do this by finding the projection of \([4, 5, 1, 7]^T\) into the subspace \(U\) of \(\mathbb{R}^4\) spanned by \([2, 1, 1, -3]^T\) and \([3, -1, 2, 4]^T\).
Here is how to find that projection (without going through the complicated process of making the basis orthonormal). We are going to pretend we have the projection (as a linear combination of the two column vectors that span $U$) and find equations that the coefficients have to satisfy, then use those equations to find the coefficients.

The projection vector $p = \text{Proj}_U(v)$ is a some linear combination of the basis vectors for $U$, that is

$$p = x \begin{bmatrix} 2 \\ 1 \\ 1 \\ -3 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \end{bmatrix},$$

where $x$ and $y$ are coefficients we want to find. Note that

$$p = M \begin{bmatrix} x \\ y \end{bmatrix}. \quad (1)$$

Let $n = v - p$. We have observed before that if we subtract from $v$ its projection to some subspace, then what is left is orthogonal to that subspace. So $n$ is orthogonal to both columns of the matrix

$$M = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 2 \\ -3 & 4 \end{bmatrix}.$$

We can say the same thing in terms of the transpose of $M$, and then do some manipulations of the matrix equation:

$$\begin{bmatrix} 2 & 1 & 1 & -3 \\ 3 & -1 & 2 & 4 \end{bmatrix} (v - p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow M^T v = M^T p$$

$$\Rightarrow \text{(using equation (1))} M^T v = M^T M \begin{bmatrix} x \\ y \end{bmatrix}.$$

So there we have our approach: We cannot solve the system

$$M \begin{bmatrix} x \\ y \end{bmatrix} = v,$$

instead we solve the so-called “normal system” associated with the given system

$$M^T M \begin{bmatrix} x \\ y \end{bmatrix} = M^T v.$$

[End of handout]