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Exam III
SOLUTIONS

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Print name ____________________ Signature ____________________

Instructions This is a "closed-book" exam. You should have no books, papers, or devices for calculation or communication available during the exam.

If you need extra space for scratch work, please use the backs of pages in the exam booklet or the occasional blank spaces in the booklet.

For all problems (except if they say "Answer Only") , you need to show enough work to justify your answers.

The test is worth 100 points. The value of each question is indicated.

1. (5 pt) _____ 2. (5 pt) _____ 3. (10 pt) _____ 4. (5 pt) _____ 5. (10 pt) _____

6. (5 pt) _____ 7. (10 pt) _____ 8. (10 pt) _____ 9. (5 pt) _____ 10. (10 pt) _____

11. (10 pt) _____ 12. (10 pt) _____ 13. (5 pt) _____

Total (100 pt) _______

Problem 1. . *(5 pts)*

Find the cosine of the angle between the vectors $(3,2)$ *and* $(1,4)$ *in* $\mathbb{R}^2$*.*

$cos(\theta) = \frac{(3,2)\cdot(1,4)}{\|(3,2)\|\ \|(1,4)\}} = \frac{11}{(\sqrt{13}\sqrt{17})}$

Problem 2. . *(5 pts)*

Find a unit vector orthogonal to $(1,2,3)$*.*

There are many vectors orthogonal to the given one. Pick any one and normalize it. For example, let $\mathbf{v} = (3, 0, -1)$. We see the dot product is zero. Now normalize $\mathbf{v}$ to get an answer: $\mathbf{u} = (3/\sqrt{10}, 0, -1/\sqrt{10})$.

Problem 3. . *(10 pts)*

Let $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ -1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$ *and* $\mathbf{u}_2 = \begin{bmatrix} -2/5 \\ 1/5 \\ 4/5 \\ 2/5 \end{bmatrix}$. *The set* $\mathcal{B} = \{\mathbf{u}_1\ ,\ \mathbf{u}_2\}$ *is an orthonormal set.*

Also the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 2 \\ 6 \end{bmatrix}$ *lies in the span of* $\mathcal{B}$*. Find the coordinates of* $\mathbf{v}$ *relative to* $\mathcal{B}$*.*

Since $\mathcal{B}$ is an orthonormal set, the coordinates of a vector are just the dot products of the vector with the two basis vectors.

$$\mathbf{v} \cdot \mathbf{u}_1 = 20/\sqrt{10}$$

$$\mathbf{v} \cdot \mathbf{u}_2 = 5$$

So, in terms of $\mathcal{B}$, we would write $\mathbf{v}$ as $(\frac{20}{\sqrt{10}}, 5)$.

Problem 4. . *(5 pts)*

Suppose $\mathbf{A}$ *is an orthogonal* 3×3 *matrix and* $\mathbf{B}$ *is an orthogonal* 3×3 *matrix. Show that the product* $\mathbf{AB}$ *also must be an orthogonal matrix.*

Hint: The transpose of a matrix product is the product of the transposes, with the ordering reversed; that is, $(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T$.

We want to show that $(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. But $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ so

$$(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{B}^T\mathbf{A}^T\mathbf{AB} = \mathbf{B}^T\mathbf{IB} = \mathbf{B}^T\mathbf{B} = \mathbf{I}$$

as desired.

Problem 5. . *(10 pts)*

Find the projection of the vector $\mathbf{v}$ *into the subspace with basis* $\mathcal{B}$*, where*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} \right\}$$

State clearly any special property or properties of $\mathcal{B}$ *that you are using to justify your method.*

The vectors in $\mathcal{B}$ are orthogonal, so we can find the projection of a vector into this subspace simply by projecting the vector onto each of the given orthogonal basis vectors and adding those two projections. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ denote the two given basis vectors. Then

$$Proj_{\mathbf{u}_1}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} \right) \mathbf{u}_1 = (3/2) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$Proj_{\mathbf{u}_2}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \right) \mathbf{u}_1 = (5/22) \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

$$Proj_{\text{the subspace}}(\mathbf{v}) = (3/2) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (5/22) \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 5/11 \\ 24/11 \\ 9/11 \end{bmatrix}$$

(Note: You can leave the answer as the above linear combination, just so it's clear that the answer is a vector [and which vector, of course]).

Problem 6. . *5 pts*

The vectors $\mathbf{u}_1 = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^T$ *and* $\mathbf{u}_2 = \begin{bmatrix}0 & 2 & 0 & 1\end{bmatrix}^T$ *span a 2-dimensional subspace, call it* $\mathcal{U}$*, of* $\mathbb{R}^4$*. For the vector* $\mathbf{v} = \begin{bmatrix}4 & -8 & 0 & 1\end{bmatrix}^T$*, the projection of* $\mathbf{v}$ *into* $\mathcal{U}$ *is*

$$proj_{\mathcal{U}}\mathbf{v} = 2\mathbf{u}_1 - 3\mathbf{u}_2 \ .$$

Find the shortest distance from the point $(4, -8, 0, 1)$ *to* $\mathcal{U}$*.*

Take the difference between $\mathbf{v}$ and its projection into $\mathcal{U}$, to get a vector $\mathbf{N}$ that is normal to the subspace $\mathcal{U}$. The length of that normal vector $\mathbf{N}$ is the distance from $\mathbf{v}$ to $\mathcal{U}$.

$$\mathbf{N} = \mathbf{v} - (2\mathbf{u}_1 - 3\mathbf{u}_2) = \begin{bmatrix}4 \\ -8 \\ 0 \\ 1\end{bmatrix} - 2\begin{bmatrix}1 \\ 0 \\ 1 \\ 0\end{bmatrix} + 3\begin{bmatrix}0 \\ 2 \\ 0 \\ 1\end{bmatrix} = \begin{bmatrix}2 \\ -2 \\ -2 \\ 4\end{bmatrix}.$$

So the distance is the length $\|\mathbf{N}\| = \sqrt{4+4+4+16} = \sqrt{28}$.

Problem 7. . *(10 pts)*

Find an orthonormal basis for the subspace of $\mathbb{R}^3$ *spanned by the vectors*

$$\mathbf{u}_1 = \begin{bmatrix}1 \\ 1 \\ 0\end{bmatrix} \ \textit{and} \ \mathbf{u}_2 = \begin{bmatrix}0 \\ 1 \\ 1\end{bmatrix} \ .$$

NOTE: To save space, I will write all the vectors here as row-vectors.

Step 1: Normalize $\mathbf{u}_1$ to get new (and final) $\mathbf{u}_1$. New $\mathbf{u}_1 = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}$.

Step 2. Project $\mathbf{u}_2$ into the subspace spanned by $\mathbf{u}_1$ and subtract that projection from $\mathbf{u}_2$. $new\mathbf{u}_2 = \mathbf{u}_2 - \text{Proj}_{\mathbf{u}_1}\mathbf{u}_2 = \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = \begin{bmatrix}0 & 1 & 1\end{bmatrix} - (1/\sqrt{2})\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\end{bmatrix} - \begin{bmatrix}1/2 & 1/2 & 0\end{bmatrix} = \begin{bmatrix}-1/2 & 1/2 & 1\end{bmatrix}$.

Now normalize $new\mathbf{u}_2$ to get the final $\mathbf{u}_2 = \begin{bmatrix}-1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}\end{bmatrix}$.

Problem 8. . *(10 pts)*

Set up, but do not solve, a system of linear equations that we could solve in order to get the **"best" approximate** *solution to the system of equations*

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - 3x_2 &= 2 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

Leave your answer in the form of an augmented coefficient matrix that a person could row-reduce in order to find the desired $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Write the given system in the form $\mathbf{Mx} = \mathbf{b}$, then multiply both sides by $\mathbf{M}^T$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 \\ -3 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Answer $\left[\begin{array}{cc|c} 6 & -3 & 8 \\ -3 & 14 & 1 \end{array} \right]$

Problem 9. . *(5 pts) Evaluate the determinant*

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 5 \end{vmatrix}$$

$$2\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} =$$

$$2(5-6) - 0 + 1(0-1) = -3$$

Problem 10. . *(10 pts)*

Evaluate each of these determinants. (Hint: Use appropriate theorems about determinants to avoid having to do a lot of arithmetic.)

(a)

$$\begin{vmatrix} 2\times 10^{-3} & 0 & 0 & 0 & 0 \\ 0 & 10^{-2} & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2\times 10^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

The determinant of a diagonal matrix is just the product of the diagonal entries, so here

$$(2\times 10^{-3})(10^{-2})(3)(2\times 10^{-1})(1) = 12\times 10^{-6} \text{ or } 1.2\times 10^{-5}$$

(b)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 6 & 2 & 2 & 2 \end{vmatrix}$$

The determinant equals 0. There are many ways to see this. For example,
Row4 $= 2\times$ Row2.

Problem 11. . *(10 pts)*

Find an eigenvector for the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 6 \\ -3 & -2 & -2 \end{bmatrix}$ *corresponding to the eigenvalue* $\lambda = 3$.

To solve $\mathbf{Mx} = \lambda\mathbf{x}$ with $\lambda = 3$, subtract 3 from each diagonal entry of $\mathbf{M}$ to get the (un-augmented) coefficient matrix of a homogeneous linear system whose solution is the desired set of eigenvectors. So we want to solve the system represented by the augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 3-3 & 0 & 0 & 0 \\ 4 & 5-3 & 6 & 0 \\ -3 & -2 & -2-3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T$ is an eigenvector for $\lambda = 3$ (or your answer can be the negative, or any other single multiple of that $\mathbf{x}$).

Problem 12. . *(10 pts) Find all the eigenvalues of the matrix*

$$\begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Step 1. Set up the determinant equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.
Step 2. Evaluate the determinant to get the characteristic polynomial, then find the roots.

$$\begin{vmatrix} 3-\lambda & 3 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} \overset{\text{(expand via column 3)}}{==} (2-\lambda)\begin{vmatrix} 3-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 5\lambda + 3)$$

We see $\lambda = 2$ is one root. For the others, use the quadratic formula to write the roots of $\lambda^2 - 5\lambda + 3$, that is $\lambda = \frac{5\pm\sqrt{13}}{2}$. So the three eigenvalues are $\left\{2,\ \frac{5}{2} + \frac{1}{2}\sqrt{13},\ \frac{5}{2} - \frac{1}{2}\sqrt{13}\right\}$.

Problem 13. . *(5 pts)*

Suppose we have a 3×3 matrix $\mathbf{M}$ that has 3 different (real, nonzero)eigenvalues, call them $\lambda_1, \lambda_2, \lambda_3$. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be eigenvectors corresponding to each of the eigenvalues. Let $\mathbf{P}$ be the 3×3 matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since the eigenvalues are different, the eigenvectors are linearly independent, so $\mathbf{P}$ has rank 3 and is invertible. Since $\mathbf{M}\mathbf{v}_1 = \lambda_1\mathbf{v}1$ etc., it follows that the matrix product

$$\mathbf{M}\,\mathbf{P} = \mathbf{P}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \textit{and so} \quad \mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

This says that IF a 3×3 matrix $\mathbf{M}$ has three distinct eigenvalues, then there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is diagonal.

Just to see if you understand what you just read... The matrix $\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has eigenvalues $\{1, 2, 3\}$ with corresponding eigenvectors $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ respectively. **What matrix P** *can we use to diagonalize $\mathbf{M}$?*

The solution that comes directly from the given data is to let $\mathbf{P}$ be the matrix whose columns are the three eigenvectors (in any order), such as:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$