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Exam II

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ANSWERS

Instructions This is a "closed-book" exam. You should have no books, papers, or devices for calculation or communication available during the exam.

If you need extra space for scratch work, please use the backs of pages in the exam booklet or the occasional blank spaces in the booklet (such as on page 3).

For all problems (except if they say "Answer Only") , you need to show enough work to justify your answers.

The test is worth 120 points. The value of each question is indicated.

1. (5 pt) _____ 2. (10 pt) _____ 3. (10 pt) _____ 4. (20 pt) _____ 5. (20 pt) _____

6. (20 pt) _____ 7. (5 pt) _____ 8. (20 pt) _____ 9. (5 pt) _____ 10. (5 pt) _____

Total (120 pt) _______

Percent (score/120) _______

Problem 1. . *(5 pts)*

Show that this subset $\mathcal{U}$ of $\mathbb{R}^2$ is NOT a *subspace.*

$$\mathcal{U} = \{(u_1, u_2) \mid u_2 = 1\}.$$

The set $\mathcal{U}$ is a straight line, but not a subspace. There are many ways to show this. Here are some:

Solution 1: $(0,1)$ *is in* $\mathcal{U}$ *but the scalar multiple* $2(0,1) = (0,2)$ *is not in* $\mathcal{U}$.
Solution 2: $(0,1)$ *and* $(1,1)$ *are in* $\mathcal{U}$ *but the sum* $(0,1) + (1,1) = (1,2)$ *is not in* $\mathcal{U}$ *because the second component is* 2*, not* 1.
Solution 3: The line does not pass through $(0,0)$.

Problem 2. . *(10 pts)*

Let $\mathbf{v} = \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}$.

Let $\mathcal{U}$ be the set of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ whose components satisfy the equation $\quad 3x_1 - 4x_2 + 5x_3 = 0.$

(a) (4pts) Check that $\mathbf{v}$ is in $\mathcal{U}$.

(b)(6pts) Check that every linear combination $a\mathbf{v} + b\mathbf{w}$ is in $\mathcal{U}$.

(a) For vector $\mathbf{v}$, $3x_1 - 4x_2 + 5x_3 = 3(-5/3) - 4(0) + 5(1) = -5 + 5 = 0$.

(b) $a\mathbf{v} + b\mathbf{w} = a \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-5/3)a + (4/3)b \\ b \\ a \end{bmatrix}$, so
$3x_1 - 4x_2 + 5x_3 = 3\left((-5/3)a + (4/3)b\right) - 4b + 5a = -5a + 4b + 5a - 4b = 0.$

Problem 3. . *(10 pts)*

Find all ways that the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ can be expressed as a linear combination of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. In other words, find all x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \mathbf{b} \ .$$

Consider the augmented coefficient matrix for this system:

$$A = \begin{bmatrix} 2 & 6 & 4 & 2 \\ 0 & 3 & 3 & 6 \end{bmatrix} \ .$$

Then row-reduce in 3 steps to

$$rref(A) = \begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 1 & 2 \end{bmatrix} \ .$$

From this we can read off

$$x_3 = \text{any value } t \ , \ x_2 = 2 - t \ , \text{ and } x_1 = -5 + t \ .$$

(extra blank space)

Problem 4. . *(20 pts)*

(This problem has two parts. The vectors in the second part are almost identical to those in the first part. The problem is written this way so you can save a little time doing arithmetic.)
(a) Show that the following three vectors are linearly independent.

$$\begin{bmatrix}1\\0\\2\\0\end{bmatrix} \quad \begin{bmatrix}2\\1\\4\\1\end{bmatrix} \quad \begin{bmatrix}0\\3\\0\\4\end{bmatrix}$$

Row-reduce

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\2 & 4 & 0\\0 & 1 & 4\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\0 & 0 & 0\\0 & 1 & 4\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \to \cdots \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

We see (even after operation 2 above) that the matrix has rank = 3. This implies that the 3 column vectors of the original matrix are linearly independent.

(b) Show that the following three vectors are linearly DEpendent. In particular, show how to express one of the vectors as a linear combination of the others.

$$\begin{bmatrix}1\\0\\2\\0\end{bmatrix} \quad \begin{bmatrix}2\\1\\4\\1\end{bmatrix} \quad \begin{bmatrix}0\\3\\0\\3\end{bmatrix}$$

As in part (a), row-reduce the matrix of columns (this time it is very helpful to complete reduce to RREF):

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 3\\2 & 4 & 0\\0 & 1 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -6\\0 & 1 & 3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

We see two pivot columns, and one non-pivot column; the third column in the original matrix is a linear combination of the first two (so the system of three column vectors is linearly dependent.) To express the third column as a linear combination of the other two, we can read off from the rref column matrix:

$$x_3 = t, x_2 = -3t, x_1 = 6t \implies x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Now let $t = 1$ *to get* $\mathbf{v}_3 = -6\mathbf{v}_1 + 3\mathbf{v}_2$.

Problem 5. . *(20 pts)* Answer the following questions about the matrix

$$M = \begin{bmatrix} 1 & 6 & 4 & -3 \\ 2 & 7 & 3 & -1 \\ 3 & 6 & 0 & 3 \end{bmatrix} \qquad \text{Hint:} \, rref(M) = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the dimension of the subspace $\mathcal{U}$ spanned by the columns of M ? _____
Two pivot columns in rref(M) $\implies$ *dimension of column-space* $= 2$.

(b) What is the rank of M ? _____
Two nonzero rows in rref(M) $\implies$ *(by definition) rank(M)*$= 2$.

(c) Find a basis for the subspace $\mathcal{U}$.
The first two columns of rref(M) are the pivot columns, so the first two columns of M are a basis for the column-space.

(d) For each column of M that is not part of the basis you found in part (c), show how to write that column vector as a linear combination of the basis vectors.
From rref(M), we see that the columns of M satisfy

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$$

where $x_4 =$ *any* t, $x_3 =$ *any* s , $x_1 = 2s - 3t$, *and* $x_2 = -s + t$. *That is,*

$$(2s - 3t) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-s + t) \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} + s \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$$

Plug in $s = 1, t = 0$, *and then plug in* $s = 0, t = 1$ *to get*

$$\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} .$$

Problem 6. . *(20 pts)*

Answer the following questions about the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad\qquad \text{Hint:} \, rref(M) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the row-space of M.

NOTE: The problem asks you to find "a basis". So your answer in each part must be some set of vectors.

Recall that row operations do not change the span of the rows. So the first two rows of rref(M) are a basis for the row-space of M: $\{[1, 0, -1, -2], [0, 1, 2, 3]\}$.

This does NOT say that the first two rows of M are a basis for the row-space: For this particular matrix M, notice that the second row is just a scalar multiple of the first row. However, we can use the first and third (or second and third) rows of M as a basis for the row-space of M. That is, matrix rref(M) teaches us that rank(M)=2; so any two linearly independent vectors in row(M) will be a basis. So other possible answers include

$$\{[1, 2, 3, 4], [4, 3, 2, 1]\} \quad OR \quad \{[2, 4, 6, 8], [4, 3, 2, 1]\}$$

(b) Find a basis for the null-space of M.

The null space is all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ *such that* $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

From rref(M), we see that if x_3 *is any value* s *and* x_4 *is any value* t, *then* $x_2 = -2s - 3t$ *and* $x_1 = s + 2t$ *give the solutions. So the the null-space of M is all vectors* $\mathbf{x}$ *of the form*

$$\begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

We have the null space of M as the set of all linear combinations of these two linearly independent vectors. So they are a basis for null(M).

Answer:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

<u>Remark</u> You will get some part credit for part (b) if you at least say that the basis has to be a set consisting of two four-dimensional vectors.

Problem 7. . *(5 pts)*

Find a matrix A that represents the given linear transformation:

$$T(x_1, x_2) = (x_1 + x_2, 2x_2, 3x_1 - x_2)$$

The matrix representing T is the matrix whose columns are the images under T of the "standard" basis vectors for the domain of T.

$$T\ (\ (1,0)\ ) = (1,0,3) \quad \textit{and} \quad T\ (\ (0,1)\ ) = (1,2,-1)\ , \textit{which says}$$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -1 \end{bmatrix}.$$

Problem 8. . *(20 pts)*

Consider the linear transformation $T : \mathbb{R}^p \to \mathbb{R}^q$ given by the matrix

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) In this example, $p =$ _____ and $q =$ _____.

Answer: $p = 3$ *and* $q = 2$

(b) The rank of the given matrix is 2. Use the "rank-nullity" theorem to calculate the dimension of the kernel of T.

The rank is 2, *and the dimension of the domain is* 3, *so "rank + nullity = dimension of domain" says nullity* $= 1$.

(c) Find a basis for the kernel of T.

This is the same as finding (a basis for) the null-space of the matrix that represents TA. We need to find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *such that*

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix, call it M, is almost row-reduced. Finish by dividing the top row by 2 to get

$$rref(M) = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{bmatrix},$$ *(continued next page)*

from which we see that if x_3 is any number t, then $x_1 = (-3/2)t$ and $x_2 = -t$ give the solutions. So *ker*$(T) =$ *all vectors* $\begin{bmatrix} (-3/2)t \\ -t \\ t \end{bmatrix}$, *and a basis for this subspace would be the set consisting of any one nonzero vector in it, for example* $\left\{ \begin{bmatrix} -3/2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(d) Give an example of two NON-zero vectors $\mathbf{v}, \mathbf{w}$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

There are many ways to do this. So in grading your exam, the issue was not whether you got the same answer as I did, but whether your own answer was valid.

Start with $\mathbf{v} =$ *any nonzero vector in the domain. If we let* $\mathbf{w} = \mathbf{v} + ($ *any vector in ker*(T) $)$, *then* $T(\mathbf{v}) = T(\mathbf{w})$. *So, in particular, we can use*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3/2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}.$$

If you start with a different $\mathbf{v}$, *you'll get a different* $\mathbf{w}$. *The point is that your vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *must differ by an element of ker*(T).

A DIFFERENT SOLUTION: Just take two different (nonzero) values of t *in the previous description of ker*(T). *For example, letting* $t = 1$ *and* $t = -1$ *gives answers*

$$\mathbf{v} = \begin{bmatrix} -3/2 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 3/2 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 9. . *(5 points)* For the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, calculate

(a) $\|\mathbf{v} + \mathbf{w}\|$

$$\|\mathbf{v} + \mathbf{w}\| = \| \begin{bmatrix} 1+2 \\ 3+3 \\ 1+4 \end{bmatrix} \| = \sqrt{3^2 + 6^2 + 5^2} = \sqrt{70}\,.$$

(b) the dot product $\mathbf{v} \cdot \mathbf{w}$.

$$\mathbf{v} \cdot \mathbf{w} = (1 \times 2 + 3 \times 3 + 1 \times 4) = 2 + 9 + 4 = 15\,.$$

Problem 10. . *(5 points)*

Decide whether or not the following matrix is an "orthogonal matrix". (Be sure to explain your reasoning.)

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\,.$$

There are many ways to see that M is NOT an "orthogonal" matrix. For example, any of the following arguments/calculations shows this:

- *Column 3 is not a unit vector.*
- *Columns 1 and 3 are not orthogonal (the dot product is 1, not 0).*
- *Columns 2 and 3 are not orthogonal (the dot product is -2, not 0).*
- *The product of M and the transpose of M is not the identity matrix:*

$$MM^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}\,.$$