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22M:033
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What we can learn from RREF(M)

This is a summary of the kind of information we can learn from the reduced row-echelon form of a matrix.

Example:

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> with(linalg):
> M:=matrix([[1,2,3,4],[5,6,7,8], [9,10,11,12],[13,14,15,16],[17,18,19,20]]);
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$$M := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix} \quad (1)$$

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> rref(M);
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$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Rank of M: We count two non-zero rows in RREF(M) and say that the "rank" of M [more precisely, the "row-rank"] is 2.

Row space of M: When you do row operations on a matrix, you do not change the row-space (the vector space that is the span of the rows). So the row space of M [Since M has 5 rows, initially all we know about the row-space is that it is the span of some set of 5 vectors.] is actually the span of two vectors, [1,0,-1,-2], and [0,1,2,3]. Furthermore, because of the [1,0],[0,1] pattern in the first two coordinates of those two rows, we see that those vectors are linearly independent. So the first two rows of RREF(M) are a basis for the row-space of M. The dimension of the row-space of M is 2.

Column space: The column space of M is the span of four vectors (namely the four column vectors of M). When we do row-operations on M, *we change the column space*. However, we still can learn information about the column space by looking at RREF(M). The columns of M that have the leading 1's in RREF(M) [say that again...We look in RREF(M) to see where are the leading 1's. The columns of RREF(M) that have the leading 1's tell us which columns of M we will call the "pivot columns" of M. In this example, there are two pivot columns and two "free variable" columns. The "free variable" columns of M can each be written as linear combinations of the pivot columns of M. And the pivot columns of M are linearly independent. Thus the pivot columns of M are a basis for the column space of M.

Null space: The null space of M is the set of all column vectors X = [x1, x2, x3, x4] (please write this as a column) for which M*X = zero vector. From the same RREF(M), we see that vectors in the null space of M must have
x1 = x3+ (2)x4 , x2 = -(2)x3 - (3)x4, x3 = any value s, and x4 = any value t. Intuitively, there are 2 "degrees of freedom" in specifying vectors in Null(M). More precisely, if we let s=0 and t=1 and x1, x2 be whatever they are supposed to be in terms of s and t, we get a vector W1 = [x1,x2, 0,1] in Null (M). Similarly, there is a vector of form W2 = [y1,y2,1,0] in Null(M). Any other vector in Null(M) is a linear combination of these two, and (because of the [1,0], [0,1] part of the W's) the vectors {W1, W2} are linearly independent. Thus {W1, W2} is a basis for Null(M). In particular, the number of free variable columns of RREF(M) tells us the dimensionof Null(M), that is what we call the "nullity" of M.

Kernel of a linear transformation: If we interpret multiplication by M as a function from R^4 to R^5, then Null(M) is the "kernel" of M. The *image* of M is the column space of M. We saw before that we can read off the dimension of the column space of M as being the same as the row-rank (i.e. the number of pivot columns). So the kernel of this linear transformation has dimension = nullity(M) = number of free-variable columns; and the image of this linear transformation has dimension = rank of M. Also, since nullity(M) > 0 (i.e. there ARE some nonzero vectors in the kernel), we know the function M(X): R^4 --> R^5 is not 1-1. In fact, just as the vector **0** has a 2-dimensional pre-image (that is, the set of vectors in R^4 that are taken to **0** by the function is a dimensional subspace of R^4), so to each point in the image of the function has a 2-dimensional preimage.

######end of handout #######

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