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Summer 06
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Handout 1
EXAMPLES OF SOLVING SYSTEMS OF LINEAR EQUATIONS

Example 1.

$$
\begin{aligned}
2x_1 + 3x_2 - 5x_3 &= 10 \\
3x_1 + 5x_2 + 6x_3 &= 16 \\
x_1 + 5x_2 - x_3 &= 10
\end{aligned}
$$

Step 1. Write the augmented coefficient matrix.
(note: I am using Matlab to keep track of this, and Matlab does not put the brackets around the matrix.)

```
A =
    2    3   -5   10
    3    5    6   16
    1    5   -1   10
```

Step 2. Use "elementary operations" on the equations, i.e. on the rows of the matrix A, to put the system into "echelon form". This "step" actually takes several smaller steps. At each stage, we replace one of the equations (i.e. replace one of the rows of matrix A) by some combination of rows: either just a rescaling of some row (i.e. some equation) or replace a row by itself plus/minus a multiple of another row.

Note: The text is happy to get each pivot element to be non-zero. I prefer making each pivot element actually = 1. This takes a bit more arithmetic, but (I think) makes it easier to read off the answers.

Step 2.1 Replace Row 1 by (1/2)*Row 1, in order to get the entry in position (1,1) to be = 1.

```
A(1,:)=(1/2)*A(1,:)

A =

    1       3/2     -5/2      5
    3       5        6       16
    1       5       -1       10
```

Step 2.1 Replace Row 2 by (Row 2 – 3*Row 1). This is the first of two changes that will put 0's under A(1,1). In other words, these two replacements together "eliminate" the variable x_2 from Equations 2 and 3.

A(2,:) = A(2,:)-3*A(1,:)

A =

1	3/2	-5/2	5
0	1/2	27/2	1
1	5	-1	10

Step (2.2) Now replace the 3rd row by (R3 – R1).

A(3,:) = A(3,:) - A(1,:)

A =

1	3/2	-5/2	5
0	1/2	27/2	1
0	7/2	3/2	5

We now have finished Column 1 of the matrix. The "pivot" entry is (+1) and the entries below that are all 0. Whatever future manipulations we do, the first column will never change.

Step (2.3) Replace Row 2 by (2*R2).

A(2,:)=2*A(2,:)

A =

1	3/2	-5/2	5
0	1	27	2
0	7/2	3/2	5

Step (2.4) Replace the 3rd row by (R3 – (7/2)* R2)

A(3,:)=A(3,:)-(7/2)*A(2,:)

A =

1	3/2	-5/2	5
0	1	27	2
0	0	-93	-2

Step (2.5) Replace row 3 by (-1/93)*R3

A(3,:)=(-1/93)*A(3,:)

A =

1	3/2	-5/2	5
0	1	27	2
0	0	1	2/93

Step (3) Once the augmented coefficient matrix has been put into "echelon form" [with JS's preference for 1's on the diagonal, or with just nonzero entries as in the text], then it is easy to read off the solution (if any exists) of the system:

$x_3 = 2/93$

$x_2 = 2- (27)(x_3) = 2-54/93 = 132/93$

$x_1 = 5 - (3/2)x_2 + (5/2)x_3 = 272/93$

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Example 2.

$$2x_1 + 3x_2 - 5x_3 = 10$$
$$3x_1 + 4x_2 + 6x_3 = 15$$
$$x_1 + x_2 + 11x_3 = 5$$

Step 1. Write the augmented coefficient matrix.

B =

2	3	-5	10
3	4	6	15
1	1	11	5

Steps (2.1), (2.2) Use two elementary row operations to get B(1,1)=1 and B(2,1)=0.

B =

1	3/2	-5/2	5
0	-1/2	27/2	0
1	1	11	5

Step (2.3) Use one elementary row operation to get B(3,1)=0

B =

1	3/2	-5/2	5
0	-1/2	27/2	0
0	-1/2	27/2	0

Step (2.4) Use one elementary row operation to get B(2,2)=1.

B =

1	3/2	-5/2	5
0	1	-27	0
0	-1/2	27/2	0

<u>Step 2.5</u> One more elementary row operation makes B(3,2)=0.

B(3,:)=B(3,:)+(1/2)*B(2,:)

B =

1	3/2	-5/2	5
0	1	-27	0
0	0	0	0

The third row of this matrix says nothing, just zero equals zero. Row 2 says
$x_2 = 27\ x_3$ and the first equation says $x_1 = 5 - (3/2)x_2 + (5/2)\ x_3 = 5 - 38\ x_3$.

So for this system, there are infinitely many solutions. There is "one degree of freedom" in that we can pick any value for x_3 , and then the values for x_2 and x_1 are determined from that.

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Example 3.

$$2x_1 + 3x_2 - 5x_3 = 10$$
$$3x_1 + 4x_2 + 6x_3 = 15$$
$$x_1 + x_2 + 11x_3 = 8$$

Here the augmented coefficient matrix is
C =

2	3	-5	10
3	4	6	15
1	1	11	8

and when we do the row-reduction process to get the matrix in echelon form, we get

C =

1	3/2	-5/2	5
0	1	-27	0
0	0	0	3

Notice the bottom row is the equation "$0\ x_1 + 0\ x_2 + 0\ x_3 = 3$". Since "0=3" is impossible, we conclude that the original system is *inconsistent*: there are no solutions.

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REDUCED ROW-ECHELON FORM OF A MATRIX (Section 1.2)

Once we get a matrix into "[row-]echelon", we can do more row operations to get an equivalent matrix (i.e. the systems of equations are equivalent – they have exactly the same solution sets) that is even simpler. Use the 1's on the diagonal to eliminate coefficients *above* them. Let's see what happen in the three cases presented above.

<u>Example 1 (continued)</u>
A =

1	3/2	-5/2	5
0	1	27	2
0	0	1	2/93

Column 1 is perfect. There is a "1" in position (1,1) and only 0's below it.

In column 2, there is a 0 below the (2,2) position, but there is a nonzero element above. Apply the row operation (R1 – (3/2)R2) --> R1.
A =

1	0	-43	2
0	1	27	2
0	0	1	2/93

Now Column 2 is perfect. Note that when we did this last operation, we did not "damage" column 1. That is because the element A(2,1) = 0 so taking A(1,1)-(3/2)A(2,1) leaves A(1,1) unchanged.

Next subtract multiples of Row 3 from Rows 1 and 2 to get zero's above the 1 in column 3.

A(1,:) = A(1,:)+43*A(3,:)

A =

1	0	0	272/93
0	1	27	2
0	0	1	2/93

A(2,:)=A(2,:)-27*A(3,:)

A =

1	0	0	272/93
0	1	0	44/31
0	0	1	2/93

So long as we are only using “elementary row operations”, we do not change the solution set of the linear system. But in the final form above, the solution is displayed as clearly as can be:
x_1 = 272/93 , x_2 = 44/31 (I left 132/93 unreduced in the first version), and x_3 = 2/93.

There is a Matlab (also Maple and probably other computer packages as well) command called **rref** that will take a matrix and put it into the final form (***r**educed **r**ow-**e**chelon **f**orm)* above.

A = [2 3 -5 10; 3 5 6 16; 1 5 -1 10]

A =

2	3	-5	10
3	5	6	16
1	5	-1	10

rref(A)

ans =

1	0	0	272/93
0	1	0	44/31
0	0	1	2/93

If we apply this to the original matrices B [that was infinitely many solutions] and C [no solutions], we get those conclusions even more clearly:

B=[2 3 -5 10; 3 4 6 15; 1 1 11 5]

B =

2	3	-5	10
3	4	6	15
1	1	11	5

>> rref(B)

ans =

1	0	38	5
0	1	-27	0
0	0	0	0

(which tells us how to express x_1 and x_2 each in terms of x_3)

and

C = [2 3 -5 10; 3 4 6 15; 1 1 11 8]

C =

2	3	-5	10
3	4	6	15
1	1	11	8

>> rref(C)

ans =

1	0	38	0
0	1	-27	0
0	0	0	1

where the last line (0=1) tells us there are no solutions.

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End of Handout 1
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