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Summer 06
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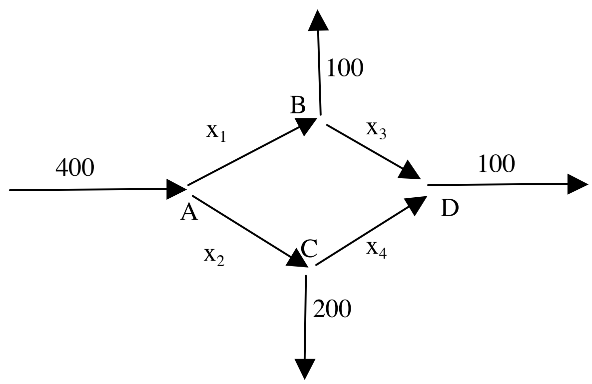
Handout 2
EXAMPLE OF NETWORK FLOW ANALYSIS

Here is a simple network (of pipes? roads? conduits? phone lines? etc.)



Assuming the system is in "steady state", with no net flow in or out of the network, find the flows x_1, x_2, x_3, and x_4.

Solution: The condition of "steady state" says that the flow in each conduit is constant, and there is no net flow in or out of the network, and no net flow in or out of any node.
This gives a system of linear equations with variables x_1, x_2, x_3, and x_4.

Node A	$x_1 + x_2 = 400$
Node B	$x_1 - x_3 = 100$
Node C	$x_2 - x_4 = 200$
Node D	$x_3 + x_4 = 100$

To solve this system of linear equations, write the augmented coefficient matrix and reduce it (using Elementary Row Operations) to reduced row-echelon form.

A =

1	1	0	0	400
1	0	-1	0	100
0	1	0	-1	200
0	0	1	1	100

rref(A)

ans =

1	0	0	1	200
0	1	0	-1	200
0	0	1	1	100
0	0	0	0	0

The reduced matrix tells us that the system has one degree of freedom, which we can represent by using x_4 as a free parameter. In terms of x4, the other flows are
$x_1 = 200 - x_4$,
$x_2 = 200 + x_4$, and
$x_3 = 100 - x_4$.

So, for example, if the conduits had valves and we open or close the valve on conduit CD to have $x_4 = 50$, then $x_1 = 150$, $x_2 = 250$, and $x_3 = 50$. If we close that valve more so as to have $x_4 = 10$, then $x_1 = 190$, $x_2 = 210$, and $x_3 = 90$.

Instead of specifying one of the flows to eliminate the degree of freedom in the system, we might want to impose some kind of symmetry (balance) condition, such as assuming that the amount of flow that goes from A to D "via" B is the same as the amount that goes from A to D "via" C. That is, assume $x_1 + x_3 = x_2 + x_4$.

With this extra equation, we get the (new) augmented coefficient matrix
B =

1	1	0	0	400
1	0	-1	0	100
0	1	0	-1	200
0	0	1	1	100
1	-1	1	-1	0

>> rref(B)

ans =

1	0	0	0	175
0	1	0	0	225
0	0	1	0	75
0	0	0	1	25
0	0	0	0	0

This says x1 = 175, x2 = 225, x3 = 75, x4 = 25 is the unique solution that satisfies the requested balance condition.

Another "practical" question we might ask is what happens if one of the conduits (say BD) gets blocked and nothing can get through it. Go back to the original matrix A (not the one with the balanced condition) and add another equation saying that $x_3 = 0$.
C =

1	1	0	0	400
1	0	-1	0	100
0	1	0	-1	200
0	0	1	1	100
0	0	1	0	0

```
>> rref(C)
```

ans =

1	0	0	0	100
0	1	0	0	300
0	0	1	0	0
0	0	0	1	100
0	0	0	0	0

Which says: In the given network, if x3 = 0, then we must have $x_1 = 100$, $x_2 =300$, $x_3=0$, $x_4=100$.

Finally, what would happen to the original system if *both* conduits AB and CD were blocked? [For a simple system like this, you can do the analysis intuitively; but for a very complicated network, this kind of question needs some math.]

The original system (represented by matrix A) now has anther row that says "$x_1 = 0$ and $x_4 = 0$".
W =

1	1	0	0	400
1	0	-1	0	100
0	1	0	-1	200
0	0	1	1	100
1	0	0	0	0
0	0	0	1	1

```
>> rref(W)

ans =

     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
     0     0     0     0     0
```

Notice the system is inconsistent: If we go back to the original network, and block conduits AB and CD, then the amount going in to node A (400) and the amount going out (200) cannot be equal.

%%%%%%%%%%%%%%End of Handout%%%%%%%%%'