22M:201 Fall 2008
Introduction to Algebraic Topology MWF 9:30
Prof. J. Simon 210 MLH

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Class notes etc. may be posted on my web page: http://www.math.uiowa.edu/~jsimon
(Office hours will be set in a few days – for now, please see me after class to make an appointment.)

Introduction:
This course will introduce some of the basic ideas of "algebraic topology" – using algebra to answer topological questions such as: Are these two spaces homeomorphic? Are these two mappings very similar to each other ("homotopic" - that will be made precise in the course)? Does a mapping $f:X \to X$ have a fixed point? Typically, the algebra can tell us that two spaces or maps are different from each other; we need more direct analysis to show they are similar. (But sometimes, if we restrict our attention to a particular set of spaces, then the algebra can provide a perfect classification.) The general method is to associate various algebraic objects (e.g. numbers, groups, rings, vector spaces) to topological spaces in such a way that similar spaces have equivalent algebraic objects. Thus, for example, a hard problem of trying to show two spaces are not homeomorphic might be changed to an easier problem of trying to show that two groups are not isomorphic.

Topologists study the "shapes" of sets; they spend half their time deciding what that means, and the other half doing it. The most(?) fundamental insight into the "shape" of a set is to count the number of components. You have studied many other topological properties of spaces: To distinguish one space from another one, we might ask if the space is compact? locally connected? separable? metric? etc. If we were trying to distinguish two smooth manifolds, we could ask about their dimensions. But what if the two spaces are both compact, connected, 2-dimensional manifolds, say a 2-sphere vs. a torus $S^1 \times S^1$. This is a typical task of Algebraic Topology. Our goal is to develop ways of associating topologically invariant numbers, groups, etc. to the spaces that will distinguish them. For example,

	$\chi(X)$ Euler Characteristic	Fundamental Group $\pi_1(X)$	First Homology $H_1(X)$	Second Homology $H_2(X)$	2^{nd} Homotopy $\pi_2(X)$
S^2	2	$\{1\}$	$\{0\}$	$\mathbb{Z}$	$\mathbb{Z}$
$S^1 \times S^1$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\{0\}$
RP^2	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

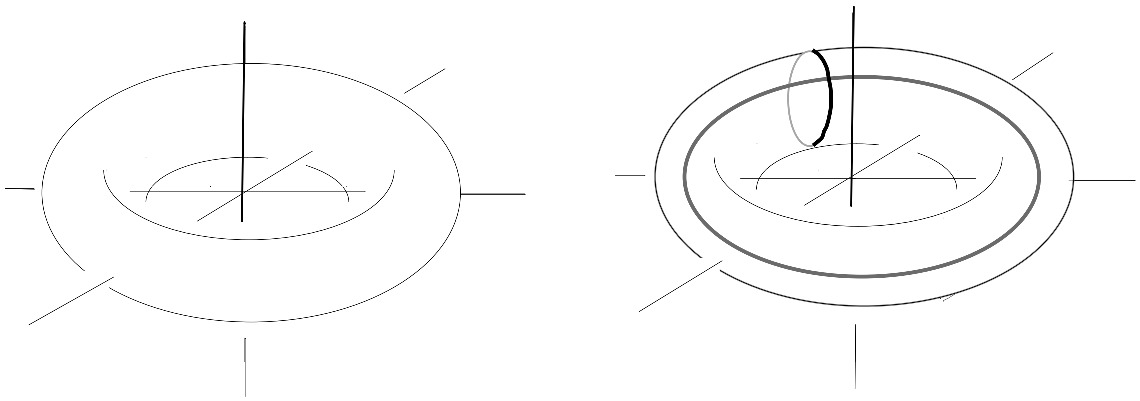
We will develop several ways to do this, with two recurring themes: (1) The algebra keeps track of what kinds of "holes" a space has, and (2) we can analyze a space by seeing how it is built out of simpler pieces. For example, the sphere S^2 is the union of two disks joined along their boundaries.

If we remove the origin from the plane $\mathbb{R}^2$, we obtain a space with a "hole" (whatever that means). We could go on deleting points from the plane and obtain spaces with any number of "holes".
The spaces $\mathbb{R}^2$ - {one point} and $\mathbb{R}^2$ - {2 points} are not homeomorphic. But it is not so easy to prove that. Both are separable metric spaces and they are identical locally - that is, each point has a neighborhood homeomorphic to an open disk. So any method that can distinguish the spaces topologically must be "aware of" the entire spaces, not just isolated parts. Furthermore, our intuition that the "number of holes" is just the number of points removed cannot be trusted completely: If we

remove an entire line segment $I^1 = \{ (x,0) \in \mathbb{R}^2 \mid 0 \le x \le 1 \}$, the resulting space is homeomorphic to what we get when we remove just one point.

Also there are different kinds of "holes". The sphere $\mathbf{S}^2$, that is the unit sphere in 3-space $\{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$ surrounds a "hole"; a standard 2-dimensional torus (see figure below) also surrounds a "hole", but the "holes" are different in some fundamental way.

Spaces with no "holes", what we might call *solid* spaces, are the simplest objects in this world of shapes. These include intervals, the real line, all cubes I^n and all Euclidean spaces $\mathbb{R}^n$.



The key idea in distinguishing the numbers and kinds of "holes" is homotopy: the ability to continuously deform one space to another (e.g. a simpler looking subspace), which we will describe in terms of continuously deforming one map to another. For example, each mapping of a circle into $\mathbb{R}^2$ can be continuously deformed to a constant map (i.e. "shrunk to a point"); but there are maps of a circle into $\mathbb{R}^2$ - {one point} that are essential, that cannot be deformed to a constant map. This teaches us that $\mathbb{R}^2$ and $\mathbb{R}^2$ - {one point} are not of the same *homotopy type*, hence are not homeomorphic.

A harder example: $\mathbb{R}^2$ - {one point} can be continuously deformed to a circle; we will say that $\mathbb{R}^2$ - {one point} and S^1 are *homotopically equivalent*, or *have the same homotopy type*. But $\mathbb{R}^2$ - {2 points} is not homotopy equivalent to S^1; which implies that $\mathbb{R}^2$ - {one point} and $\mathbb{R}^2$ - {2 points}are not homeomorphic. To show that $\mathbb{R}^2$ - {two points} is homotopically different from S^1, we invent a "multiplication" on the set of maps of a circle into a space X; somehow we can take two maps of $\mathbf{S}^1 \to X$ and produce a new map of $\mathbf{S}^1 \to X$ that combines in a meaningful way the original two maps.♥ Once we have a way to combine maps, we actually can make them into a group. In this sense, the group associated with $\mathbb{R}^2$ - {one point} is cyclic, whereas the group associated with $\mathbb{R}^2$ - {2 points} is not generated by any one element. These are the sorts of ideas involved in the *fundamental group* of a space and its natural companion, *covering spaces*.

♥ For this combining, we don't actually work with maps from a circle into *X;* we work with maps from an interval *[0,1]* where the two endpoints are sent to the same place.

Another basic idea related to "holes" is the notion of one set being the *boundary* of another. A circle in the plane is the boundary of a disk; the 2-dimensional torus (above) in $\mathbb{R}^3$ is the boundary of a solid torus. If our whole world were just the 2-dimensional torus, then we would have circles that do not bound any disks. If you have studied vector calculus, in particular Green's, Stokes's, and Gauss's theorems, then you've seen important situations where the average behavior of a function on a set can be described by its behavior on the *boundary* of the set. (e.g. if **F** is a vector field on $\mathbb{R}^3$, then the integral of the divergence of **F** over some domain is equal to the flux of **F** through the boundary of the domain.) If you were careful about stating those calculus theorems, you know there were subtle issues of orientation: the set and the boundary components had to be oriented consistently. By thinking about things that are "capable of being boundaries" (we call them *cycles*), we are led to develop *homology* theories: A space contains *cycles;* some of the cycles do not bound anything, and those are the ones that capture the holes in the space. We can invent ways to "add" cycles, and again produce groups that describe the shape of the space.

A given space X has homology groups $H_0(X)$, $H_1(X)$, etc., one *group* $H_p(X)$ for each dimension p . Actually, we also will have a lot of groups for each p since there is freedom to choose a *coefficient group*. If the coefficient group is the integers (so we may write $H_p(X;\mathbb{Z})$) then we are assigning abelian groups, i.e. $\mathbb{Z}$-modules; if the coefficient group is a field, say the reals, we write $H_p(X;\mathbb{R})$ and we are choosing a real vector space to measure the "*p-dimensional holes" in X.*

For each way of assigning groups (or vector spaces or other algebra objects) to a space, we also have a way of assigning homomorphisms to continuous maps of spaces. If $f: X \to Y$ is a continuous map of spaces, then we will associate to f a homomorphism of groups $\mathbf{f}_*: H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$. This grand machine takes time to define, and it sometimes is complicated; but the reward is a way to prove that various spaces cannot be topologically equivalent, and that various maps are not homotopic to each other. As a consequence, we obtain famous results such as the Brouwer Fixed Point Theorem and the Jordan Curve Theorem.

There is a way to combine "holes" of some dimensions to produce "holes" of different dimensions. For example, the 2-dimensional torus (shown above) has two "1-dimensional holes", corresponding to two special curves on the surface. When we "multiply" them together, we get a "2-dimensional hole"; this is a rather sophisticated algebraic process analogous to taking the Cartesian product of two sets. On the other hand, a 2-sphere has a "2-dimensional hole" that does not arise from any "1-dimensional holes". In order to make this precise, and gain the ability to distinguish spaces whose individual homology groups are identical, we go to a higher level construction, the **co**homology groups of a space. These support a multiplication between different dimensions and together form a ring. If we have time, the cohomology ring structure, and its applications to orientation of manifolds and duality will be the final topics of our course. More likely, you will study them in a future course.

The first topic in our course is *surfaces*. We want to have a library of spaces that are common in mathematics, reasonably simple, yet topologically varied enough to motivate and illustrate the methods we will develop.

Texts 1. A Basic Course in Algebraic Topology by William Massey (Academic Press, Graduate Texts #127). (Chapters I and V-IX.)

2. Algebraic Topology by Allen Hatcher. (Chapters 0, I, Appendix) Available online at http://www.math.cornell.edu/~hatcher/AT/ATpage.html.

Grading: Your grade will be based on weekly homework, two mid-term exams, a final exam, and class participation (which includes regular attendance). The weighting will be approximately:

Midterm Exam #1	20%
Midterm Exam #2	20%
Homework	30%
Final Exam	30%

I expect to use the above weights to compute a numerical average representing the minimum grade you have earned, and then "round up" or add some additional amount if your class participation has been strong; for example, a 3.40 might become an A-, or even A, this way.

Working together: I encourage you to study in groups – it helps a lot when you are trying to learn something if you can explain it to someone else. But your homework is supposed to be your own work. I realize that sometimes it is difficult to draw the line between healthy cooperation and plagiarism, but we all have to be careful about keeping that distinction.

Schedule:

Week of October 13: Midterm Exam #1 7-8:30 p.m. Day and Room to be announced.

Week of November 17: Midterm Exam #2 7-8:30 p.m. Day and Room to be announced.

Thursday December 18 Final Exam 9:45-11:45 a.m. Room 210 MLH

Note: Monday Sept. 1 is a University holiday. Also we will not have class on Wednesday October 1.

Special notes:

This course description represents my current intentions. Changes may be announced in class or by email as needed.

If you wish to contact the Mathematics Department Chair, his office is in 14 MLH. To make an appointment, call 335-0708 or contact the Department Secretary in 14C MLH.

Please let me know if you have a disability, which requires special arrangements.

My own expectation in this course is that we will deal with each other, and with the course material, in a responsible, professional, honorable way, and that we will enjoy working together this term. I welcome your comments, good or bad, about any aspect of the course, any time during the semester, and in the student evaluation forms used at the end.

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The College of Liberal Arts and Sciences

Policies and Procedures

Administrative Home
The College of Liberal Arts and Sciences is the administrative home of this course and governs matters such as the add/drop deadlines, the second-grade-only option, and other related issues. Different colleges may have different policies. Questions may be addressed to 120 Schaeffer Hall or see the Academic Handbook.
www.clas.uiowa.edu/students/academic_handbook/index.shtml

Academic Fraud
Plagiarism and any other activities when students present work that is not their own are academic fraud. Academic fraud is reported to the departmental DEO and to the Associate Dean for Academic Programs and Services who enforces the appropriate consequences.
www.clas.uiowa.edu/students/academic_handbook/ix.shtm

Making a Suggestion or a Complaint
Students with a suggestion or complaint should first visit the instructor, then the course supervisor and the departmental DEO. Complaints must be made within six months of the incident.
www.clas.uiowa.edu/students/academic_handbook/ix.shtml#5

Accommodations for Disabilities
A student seeking academic accommodations should register with Student Disability Services and meet privately with the course instructor to make particular arrangements. For more information, visit this site:
www.uiowa.edu/~sds/

Understanding Sexual Harassment
Sexual harassment subverts the mission of the University and threatens the well-being of students, faculty, and staff. Visit www.sexualharassment.uiowa.edu for definitions, assistance, and the full University policy.

Reacting Safely to Severe Weather
In severe weather, the class members should seek shelter in the innermost part of the building, if possible at the lowest level, staying clear of windows and free-standing expanses. The class will continue if possible when the event is over. (Operations Manual 16.14. i.)

Important University of Iowa Deadlines for Off-Cycle Courses
(Note: Use only for off-cycle courses. To find the deadlines for a particular course, visit this Registrar site and type in the course number and title: www.registrar.uiowa.edu/more/coursedeadlines.aspx)
Since this course begins or ends at a time different from other courses, please be aware of these deadlines:
Last day to add:
Last day to drop:

Resources
_ Writing Center 110 English-Philosophy Building, 335-0188, http://www.uiowa.edu/ewritingc
_ Speaking Center 12 English-Philosophy Building, 335-0205, http://www.uiowa.edu/erhetoric/centers/speaking
_ Mathematics Tutorial Laboratory 314 MacLean Hall, 335-0810, http://www.math.uiowa.edu/mathlab/index.htm
_ Tutor Referral Service Campus Information Center, Iowa Memorial Union, 335-3055, http://www.imu.uiowa.edu/cic/tutor referral service

Student Classroom Behavior

The ability to learn is lessened when students engage in inappropriate classroom behavior, distracting others; such behaviors are a violation of the Code of Student Life. When disruptive activity occurs, a University instructor has the authority to determine classroom seating patterns and to request that a student exit immediately for the remainder of the period. One-day suspensions are reported to appropriate departmental, collegiate, and Student Services personnel (Office of the Vice President for Student Services and Dean of Students).

University Examination Policies

Missed exam policy. University policy requires that students be permitted to make up examinations missed because of illness, mandatory religious obligations, certain University activities, or unavoidable circumstances. Excused absence forms are available at the Registrar web site:
http://www.registrar.uiowa.edu/forms/absence.pdf

Final Examinations

An undergraduate student who has two final examinations scheduled for the same period or more than three examinations scheduled for the same day may _le a request for a change of schedule before the published deadline at the Registrar's Service Center, 17 Calvin Hall, 8-4:30 M-F, (384-4300).

*The CLAS policy statements have been summarized from the web pages of the College of Liberal Arts and Sciences.