1. First definitions

**Definition 1** (Standard n-cells).

$I^n$ is the cube $[0, 1] \times \ldots \times [0, 1] \subset \mathbb{R}^n$.

We want $I^0$ to be a single point. Let us denote this as $I^0 = \vec{0}$ just to emphasize that $I^0$ is a point.

**Definition 2** (Singular n-cubes in a space $X$). A continuous function $T : I^n \to X$ is called a singular n-cube in $X$.

**Remark.** Informally, we can think of a singular 0-cube as a point in $X$; there are as many maps $T : I^0 \to X$ as there are points in $X$.

**Remark.** Formally, we can think of a singular 1-cube in $X$ as a path in $X$. We have studied paths a lot already, and so you are already an expert on singular 1-cubes.

**Definition 3** (Group of singular n-cubes in $X$). Let $Q_n$ denote the (abstract) free abelian group with basis = the set of all singular n-cubes in $X$.

The group $Q_n(X)$ consists of finite linear combinations of functions; two different functions do not interact at this level; but coefficients can cancel. For example, suppose $X$ is $\mathbb{R}^1$ and $S, T$ are singular 2-cubes in $X$ given by recipes

$$T(u, v) = u + v \quad \text{and} \quad S(u, v) = -u - v.$$  

Then, in $Q_2(X)$, the subgroup generated by $S$ and $T$ has rank = 2. The fact that there is some arithmetic relation between the recipes defining $S$ and $T$ does not produce a relation between $S$ and $T$ as elements of $Q_2(\mathbb{R})$ (unless the two recipes worked out to exactly the same function, which here they don’t). On the other hand, relations like $2T - 3T = -T$ are valid in $Q_2$.

**Definition 4** (Degenerate and nondegenerate cubes). A singular cube $T : I^n \to X$ is called degenerate if the function is independent of one of the coordinate values.
Remark. A singular 0-cube cannot be degenerate. A singular 1-cube is degenerate if and only if it is a constant function.

Remark (Example). In $Q_3(\mathbb{R})$, the 3-cube $T(u, v, w) = w^2(u + \cos(u - w)^3)$ is degenerate because it is independent of $v$. The singular 3-cube $T(u, v, w) = u + v + w$ is NON-degenerate, even though it is a function from a 3-dimensional cell to a 1-dimensional space.

Definition 5 (Singular n-chains, $C_n$). Since $Q_n$ is generated by all singular cubes, we can write $Q_n$ as a direct sum of the free abelian group generated by degenerate cubes and the free abelian group generated by nondegenerate cubes, which we call $C_n(X)$.

$$Q_n(X) = D_n(X) \oplus C_n(X).$$

Alternatively, we can define $C_n(X)$ as the quotient

$$C_n(X) = \frac{Q_n(X)}{D_n(X)}.$$

Technically, an element of $C_n$ is an equivalence class, or an ordered pair whose first coordinate is 0. Two elements of $C_n$ are equal if their difference is (a linear combination of) degenerate (n-cubes).

Remark (Different homology theories). There are other ways to develop homology theories: for example, one can work with triangulated spaces where the “cells” are actual segments, triangles, etc. (“simplicial homology”); one can work with continuous maps from simplices into $X$ (“singular homology” usually refers to this); and there are generalizations to spaces that are more general topologically (see Čech or Vietoris homology theories). Each approach has some technical price(s) to pay for the ways in which it is “nice”. The price we pay for singular cubical homology is that we have to fuss with these degenerate cubes.

1.1. The boundary of a singular cube. If $T : I^n \to X$ is a singular cube, we want to define the boundary of $T$ as an algebraic object (specifically, an element of $Q_{n-1}(X)$) that somehow captures our intuitive idea of boundary. Geometrically, the boundary of a cube $I^n$ is the set of its faces. If we restrict $T$ to one of the faces of $I^n$, we can define a singular $(n - 1)$-cube; and by combining these a certain way, we will get a linear combination of $2n$ singular $(n - 1)$-cubes representing the boundary of $T$.

Definition 6 (The $i^{th}$ front- and back- faces of a singular n-cube). Suppose $T : I^n \to X$ is a singular n-cube. (Assume $n \geq 1$; we will
consider the case of 0-cubes later.) Fix an index $i$, $1 \leq i \leq n$. Define
the $(n - 1)$-cube $(A^n_i T) : I^{n-1} \to X$ by

$$(A^n_i T)(s_1, \ldots, s_{n-1}) = T(s_1, \ldots, s_{i-1}, 0, s_i, \ldots, s_n).$$

Note: $T(0, s_1, \ldots, s_{n-1})$ and $T(s_1, \ldots, s_{n-1}, 0)$ are the extreme cases
$i = 1, n$. Similarly, we define

$$(B^n_i T)(s_1, \ldots, s_{n-1}) = T(s_1, \ldots, s_{i-1}, 1, s_i, \ldots, s_n).$$

We now define the boundary of a singular cube $T : I^n \to X$ as

$$\partial T = \sum_{i=1}^{n} (-1)^i (A^n_i T - B^n_i T).$$

In this way, we define the boundary of a singular n-cube to be a
certain linear combination of singular $(n - 1)$-cubes. Since the
n-cubes are a basis for the free abelian group $Q_n$, the function $\partial$
extends to a homomorphism $\partial_n : Q_n(X) \to Q_{n-1}(X)$.

**Lemma 0.1 (HOMEWORK).** If $T$ is a degenerate n-cube in $X$, then
$\partial_n(T) \in D_{n-1}(X)$.

**Remark.** Note the above lemma does not claim that each face of a
degenerate cube is degenerate; some faces are degenerate, and the
non-degenerate ones cancel.

Once we know the above lemma, that $\partial_n|_{D_n} : D_n \to D_{n-1}$, we see that
$\partial_n$ induces a homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$.

**Theorem 1.** The composition of homomorphisms

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

is the zero map. Put briefly,

$$\partial_n \circ \partial_{n+1} = 0.$$

Even more cryptically (but a nice way to remember this key parts of
any homology theory),

$$\partial \partial = 0.$$

**Proof: HOMEWORK.** To prove this theorem, you need to
understand faces-of-faces. In the next section, we show how to derive
the “face identities” that you can use to prove the theorem. (You
don’t have to prove the face identities.)
1.2. Face identities. We can view the operations of taking faces as homomorphisms

\[ A^n_i, B^n_i : Q_n(X) \rightarrow Q_{n-1}(X) . \]

What happens when we take a face-of-a-face? If \( T \) is an \( n \)-cube, then \( A^n_i T \) is an \((n - 1)\)-cube, and \( A^{n-1}_j A^n_i T \) is an \((n - 2)\)-cube. For example, if \( T \) is a singular 5-cube, then

\[ (B^4_2 A^5_2 T)(s_1, s_2, s_3) = T(s_1, 0, 1, s_2, s_3) . \]

We see this as follows: \( A^5_2 T \) is the 4-cube

\[ (t_1, t_2, t_3, t_4) \rightarrow T(t_1, 0, t_2, t_3, t_4) . \]

The \( B^4_2 \) face of of \( A^5_2 T \) is the 3-cube

\[ (s_1, s_2, s_3) \rightarrow (A^5_2 T)(s_1, 1, s_2, s_3) = T(s_1, 0, 1, s_2, s_3) . \]

If we use the dimension-indicating superscripts (as \( B^4 \) and \( A^5 \) above), we can see that \( B_i A_j \) is not the same as \( A_j B_i \), and establish the following identities:

**Proposition 1.1** (Eqns. 7.2.1, Sec. VII.2.1 of Massey).

\[
\begin{align*}
A_i A_j & = A_{j-1} A_i \\
B_i B_j & = B_{j-1} B_i \\
A_i B_j & = B_{j-1} A_i \\
B_i A_j & = A_{j-1} B_i
\end{align*}
\]

1.3. Cycles, Bounds, and Homology Groups. If \( z_n \in C_n \) such that \( \partial_n z_n = 0 \), we call \( z_n \) a \( n \)-cycle. The group of \( n \)-cycles is

\[ Z_n = \ker \partial_n . \]

The image group \( \partial_{n+1}(C_{n+1}) \subseteq C_n \) is denoted \( B_n \), the group of \( n \)-bounds. Since \( \partial_n \circ \partial_{n+1} = 0 \), we have

\[ B_n \subseteq Z_n . \]

Thus the quotient group

\[ H_n(X) = \frac{Z_n(X)}{B_n(X)} \]

makes sense; this is the \( n \)-dimensional (singular cubical) homology group of \( X \).
Remark. As with the fundamental group, it will be easy to show that homology groups are topological invariants. But it will be difficult to show that any of them are nontrivial beyond dimension 0. Be patient. Meanwhile, let’s establish a relation between $\pi_1(X)$ and $H_1(X)$. The full theorem (the Hurewicz theorem) says that for a path-connected space, $H_1(X) \cong \text{abelianized } \pi_1(X)$. In the exercise below, you are asked to establish a part of this relationship.

Proposition 1.2.

(1) Suppose $f : [0, 1] \to X$ is a loop based at $x_0 \in X$. Then, viewing $f$ as a singular 1-cube, $f$ is a cycle.

(2) Suppose the loop $f : [0, 1] \to X$ is homotopically trivial, rel endpoints, in $X$. (That is, $[f] = 1 \in \pi_1(X, x_0)$.) Then the 1-cycle $f$ is homologically trivial, i.e. $[f] = 0 \in H_1(X)$.

Proof: HOMEWORK. You should do this exercise by working directly from the careful definitions of boundary, cycle, etc. To show that an n-chain is a cycle, show that its boundary $= 0 \in C_{n-1}$; but recall that all degenerate chains are 0 in $C_{n-1}$. To show that an n-cycle is homologically trivial, you need to show that it lies in $B_n$; but, again, to “be” an element of $B_n$, the chain just needs to differ from an actual boundary by 0 or by something degenerate. □

End of Handout