Homology of a Cell Complex

A finite cell complex $X$ is constructed one cell at a time, working up in dimension. Each time a cell is added, we can analyze the effect on homology and, by this inductive process, calculate the homology groups of $X$.

In these notes, we will develop some of the main properties of homology of finite cell complexes by using this inductive approach.

Recall that each finitely generated abelian group $G$ can be expressed as the direct product $F \times T$, where $F$ is a free abelian group $\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ and $T$ is finite. The number of $\mathbb{Z}$ factors is called the rank of $G$. The rank of $G$ is well-defined; that is, given any two decompositions of $G$ as $F_1 \times T_1$ or $F_2 \times T_2$, the ranks of the free abelian parts must be the same. (However, there are different ways to express the torsion parts; for example, $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.)

**Theorem 1.** Suppose $X$ is a finite $CW$-complex of $[maximum]$ dimension $n$. For each $k \geq 0$, let $c_k$ denote the number of cells of $X$ of dimension $k$, and let $\beta_k$ denote the rank of $H_k(X)$. Then:

1. For each $k \geq n + 1$, $H_k(X) = \{0\}$.
2. $H_n(X)$ is a free abelian group of rank $\leq c_n$.
3. For each $k$, $H_k(X)$ is generated by some set of $c_k$ [or fewer] elements. In other words, $H_k(X)$ is a quotient group of the free abelian group $\mathbb{Z}^{c_k}$.
4. $\sum_{k=0}^{n} (-1)^k c_k = \sum_{k=0}^{n} (-1)^k \beta_k$.

Note that part (4) says the Euler characteristic of $X$ is a topological invariant [in fact, a homotopy-type invariant]. The alternating sum of the numbers of cells in each dimension is the same, regardless of how we triangulate $X$ or otherwise express $X$ as a $CW$-complex.

Part (3) does not say that $\beta_k = c_k$; typically, $\beta_k < c_k$. But it does imply that each $\beta_k$ is finite. This is why the sum of Betti numbers in part (4) makes sense.
Theorem 1 is a corollary of the following.

**Theorem 2.** Suppose a space $X$ is obtained [as a quotient space] by attaching a cell $B^n$ to a space $Y$ via a map $f : S^{n-1} \to Y$. ($S^{n-1}$ is the boundary sphere of $B^n$.) Then the homology groups of $X$ and $Y$ are related as follows:

- For all $k \neq n, n-1$, $H_k(X) \cong H_k(Y)$.
- Exactly one of the following must happen:
  \[ \beta_{n-1}(X) = \beta_{n-1}(Y) - 1 \quad \text{and} \quad H_n(X) \cong H_n(Y) \]
  or else
  \[ \beta_{n-1}(X) = \beta_{n-1}(Y) \quad \text{and} \quad H_n(X) \cong H_n(Y) \times \mathbb{Z} \]

**Remark.** Theorem 2 says exactly what $H_n(X)$ must be in the two cases; but it does not say exactly what $H_{n-1}(X)$ must be: it just specifies the rank $\beta_{n-1}$. For example, if $Y$ is a Möbius band, and we attach a 2-cell $B^2$ to $Y$ via a homeomorphism $f$ of their boundaries (so $X$ is $\mathbb{R}P^2$) then

\[
H_1(X) \cong \mathbb{Z}_2 \quad \beta_1(X) = 0 = \beta_1(Y) - 1 \quad H_2(X) = \{0\}.
\]

If, instead, the attaching map $f$ identifies the boundary of $B^2$ with the center-line of $Y$, then we have

\[
H_1(X) \cong \{0\} \quad \beta_1(X) = 0 = \beta_1(Y) - 1 \quad H_2(X) = \{0\}.
\]

The basic idea of Theorem 2 is that when we add a new n-cell to $Y$, we either increase the rank of $H_n$ or else decrease the rank of $H_{n-1}$. More precisely, adding an n-cell to $Y$ “kills” an element of $H_{n-1}(Y)$; if that element has infinite order in $H_{n-1}(Y)$, then we change the rank of $H_{n-1}(Y)$; if that element is zero, or has finite order, in $H_{n-1}(Y)$, then we don’t change the rank of $H_{n-1}(Y)$, but we do create a new free generator for $H_n$.

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Exam sample: Prove Thm 2 for a specific $n$, e.g $n=3$. Invoke lemmas as needed w/o pf.

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Exam sample: Outline proof of Theorem 1.

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Prove Thereom 1 (assuming Theorem 2).

**Hint:** First prove the theorem in the case $n = 0$ (and $c_n =$ all values). Let $(c_0, c_1, \ldots, c_n)$ be the list of numbers of cells of $X$. Do the proof by induction on this n-tuple. For the inductive step, Let $Y$ be all of $X$ except for the final n-cell. Then, by inductive assumption, $Y$ satisfies the theorem. Now apply Theorem 2 and work out the bookkeeping.
Proof of Theorem 2

Lemmas for Theorem 2. We need a topology lemma and an additional small algebra lemma.

Lemma 2.1. Suppose a space $X$ is obtained [as a quotient space] by attaching a cell $B^n$ to a space $Y$ via a map $f : S^{n-1} \to Y$. Then for each $k$,
\[ H_k(X, Y) \cong H_k(B^n, S^{n-1}) . \]

Proof. View $B^n$ as the unit ball in $\mathbb{R}^n$. Let $S_1 = \text{boundary} (B^n)$ and let $S_{1/2}$ be the sphere of radius $\frac{1}{2}$ centered at the origin. Let $U$ be the closed spherical shell bounded by $S_{1/2}$ and $S_1$. Let $B_{1/2}$ denote the closed ball bounded by $S_{1/2}$. Note that $S_1$ is a strong deformation retract of $U$.

When we attach the ball $B^n$ to $Y$ via map $f : S_1 \to Y$, the set $U \cup Y$ becomes the mapping cylinder of $f$. Let $Y_{1/2}$ denote this set, $U \cup f Y$. We claim that the pair $(X, Y_{1/2})$ is homotopy equivalent to the pair $(X, Y)$.

The pair of line segments $([0, 1], [\frac{1}{2}, 1])$ is homotopy equivalent to the pair $([0, 1], \{1\})$: just gradually shrink $[\frac{1}{2}, 1]$ to its right endpoint, while gradually expanding $[0, 1]$ to fill $[0, 1]$. Apply this radially to $B^n$ to get a homotopy equivalence $(B^n, U) \simeq (B^n, S_1)$. Do the same deformations, but now with the identifications made on $S_1$ via $f$, to see how to deformation retract $Y_{1/2}$ to $Y$ while expanding $B^n - \text{int}U$ to fill $B^n$. (This is just a slight modification of the proof that a mapping cylinder deforms to its target end – see Hatcher’s discussion of mapping cylinders – the target end is the easy end.) Thus we have a homotopy equivalence
\[ (Y \cup f B^n, Y \cup f U) \simeq (Y \cup f B^n, Y) , \]
i.e. $(X, Y_{1/2}) \simeq (X, Y)$.

The next step is to excise $Y$ from $(X, Y_{1/2})$. We introduced the neighborhood $U$ in order to ensure that $Y$ is contained in the interior of $Y_{1/2}$. By the Excision Theorem, for each $k$,
\[ H_k(X, Y_{1/2}) \cong H_k(X - Y, Y_{1/2} - Y) . \]

But $X - Y$ is just the open ball $B^n - S_1$, and $Y_{1/2} - Y$ is the half-open shell $U - S_1$, so $(X - Y, Y_{1/2} - Y) \simeq (B_{1/2}, S_{1/2}) \cong (B^n, S^{n-1})$.  

\[ \square \]
Lemma 2.2. Suppose $\phi : A \to B$ is a group homomorphis. Then $\phi$ defines an injection $\hat{\phi} : G / \ker \phi \to H$.

Proof. Easy unassigned exercise. 

We can use Lemma 2.2 to break a long exact sequence into a collection of short exact sequences. Suppose we have groups and homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

where $\text{im}(f) = \ker(g)$. Then

$$A \xrightarrow{f} \text{im}(f) \to \{0\}$$

and

$$\{0\} \to B / \text{im}(f) \xrightarrow{\delta} C$$

are exact.

Proof of Theorem 2. Consider the long exact sequence for the pair $(X, Y)$.

$$\cdots \to H_{k+1}(X, Y) \to H_k(Y) \to H_k(X) \to H_k(X, Y) \to \cdots .$$

By Lemma 2.1, for each $k$, $H_k(X, Y) \cong H_k(B^n, S^{n-1})$, which we know from earlier work (recall how we calculated homology groups of spheres) is isomorphic to $\tilde{H}_k(S^n)$. So in the long exact sequence, we can replace each $H_k(X, Y)$ by 0 if $k \neq n$, and $\mathbb{Z}$ for $k = n$. Now consider the cases: $k > n$, $k < n - 1$, and the intermediate situations.

Cases $k > n$ or $k < n - 1$

We have

$$\cdots \to \{0\} \to H_k(Y) \to H_k(X) \to \{0\} \to \cdots ,$$

so $H_k(X) \cong H_k(Y)$.

Cases $k = n$, $n-1$

We have $H_{n+1}(X, Y) = \{0\} = H_{n-1}(X, Y)$, so the homology groups in dimensions $n$ and $n - 1$ are connected by the finite exact sequence

$$\{0\} \to H_n(Y) \to H_n(X) \to H_n(X, Y) \to H_{n-1}(Y) \to H_{n-1}(X) \to \{0\} .$$

We also know $H_n(X, Y) \cong \mathbb{Z}$, so we actually have

$$\{0\} \to H_n(Y) \to H_n(X) \xrightarrow{j} \mathbb{Z} \to H_{n-1}(Y) \to H_{n-1}(X) \to \{0\} .$$

We have been omitting the names of the various homomorphisms, but now we need to refer to the function $j_*$. The image of $j_*$ is some subgroup $p\mathbb{Z}$ of $Z$. There are two separate cases: Either $p = 0$ or else $p \neq 0$, and these are the two different cases for Theorem 2.
Sub-case $p = 0$.
By Lemma 2.2 (see the discussion following that lemma) our sequence breaks into two short exact sequences

$$\{0\} \longrightarrow H_n(Y) \longrightarrow H_n(X) \xrightarrow{j_*} \{0\}$$

and

$$\{0\} \longrightarrow \mathbb{Z}/\{0\} \longrightarrow H_{n-1}(Y) \longrightarrow H_{n-1}(X) \longrightarrow \{0\}.$$  

This says that $H_n(X) \cong H_n(Y)$ and (by the “rank-nullity theorem” for finitely generated abelian groups) $\beta_{n-1}(Y) = \beta_{n-1}(X) + 1$, which is the first alternative in Theorem 2.

Sub-case $p \neq 0$.
Again by Lemma 2.2, the sequence breaks into short exact sequences

$$\{0\} \longrightarrow H_n(Y) \longrightarrow H_n(X) \xrightarrow{j_*} p\mathbb{Z} \longrightarrow \{0\}$$

and

$$\{0\} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow H_{n-1}(Y) \longrightarrow H_{n-1}(X) \longrightarrow \{0\}. $$

The subgroup $p\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so the first short exact sequence splits, to give $H_n(X) \cong H_n(Y) \times \mathbb{Z}$. Meanwhile, the group $\mathbb{Z}/p\mathbb{Z}$ is finite, so the rank-nullity theorem says $\beta_{n-1}(X) = \beta_{n-1}(Y)$.

This completes the proof of Theorem 2.  \qed