22M:201
Introduction to Algebraic Topology
Prof. J. Simon

Fall 2006
MWF 10:30
114 MLH

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Class notes etc. may be posted on my web page: http://www.math.uiowa.edu/~jsimon
(Office hours will be set in a few days – for now, please see me after class to make an appointment.)

Introduction:
This course will introduce some of the basic ideas of "algebraic topology" – using algebra to answer topological questions such as: Are these two spaces homeomorphic? Are these two mappings very similar to each other ("homotopic" - that will be made precise in the course)? Does a mapping $f{:}X \to X$ have a fixed point? Typically, the algebra can tell us that two spaces or maps are different from each other; we need more direct analysis to show they are similar. (But sometimes, if we restrict our attention to a particular set of spaces, then the algebra can provide a perfect classification.) The general method is to associate various algebraic objects (e.g. numbers, groups, rings, vector spaces) to topological spaces in such a way that similar spaces have equivalent algebraic objects. Thus, for example, a hard problem of trying to show two spaces are not homeomorphic might be changed to an easier problem of trying to show that two groups are not isomorphic.

Topologists study the "shapes" of sets; they spend half their time deciding what that means, and the other half doing it. The most(?) fundamental insight into the "shape" of a set is to count the number of components. You have studied many other topological properties of spaces: To distinguish one space from another one, we might ask if the space is compact? locally connected? separable? metric? etc. If we were trying to distinguish two smooth manifolds, we could ask about their dimensions. But what if the two spaces are both compact, connected, 2-dimensional manifolds, say a 2-sphere vs. a torus $S^1 \times S^1$. This is a typical task of Algebraic Topology. Our goal is to develop ways of associating topologically invariant numbers, groups, etc. to the spaces that will distinguish them. For example,

	$\chi(X)$ Euler Characteristic	Fundamental Group $\pi_1(X)$	$H_1(X)$ First Homology	$H_2(X)$ Second Homology	$\pi_2(X)$ Second Homotopy
S^2	2	{1}	{0}	$\mathbb{Z}$	$\mathbb{Z}$
$S^1 \times S^1$	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	{0}

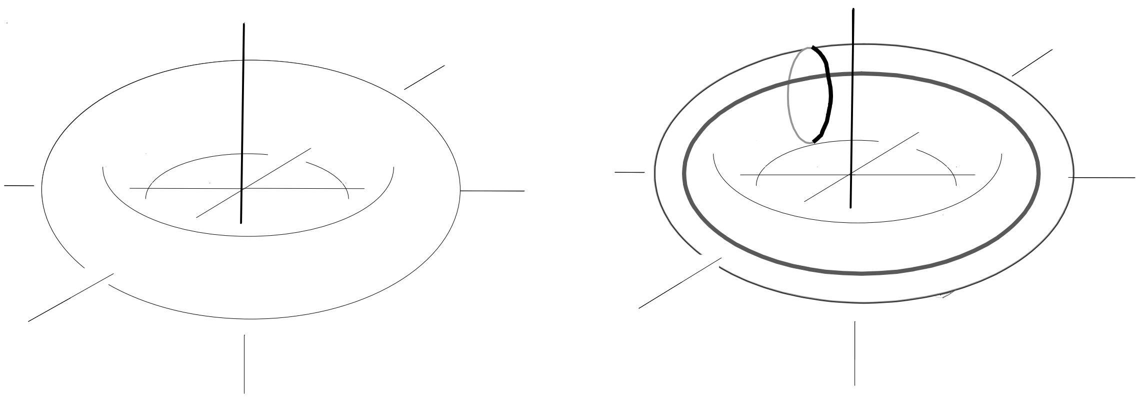
We will develop several different ways to do this, with two recurring themes: (1) We will try to understand a space by seeing how it can be built out of simpler pieces. For example, the sphere S^2 is the union of two disks joined along boundaries. (2) We will try to describe the "holes" in a space.

If we remove the origin from the plane $\mathbb{R}^2$, we obtain a space with a "hole" (whatever that means). We could go on deleting points from the plane and obtain spaces with any number of "holes".
The spaces $\mathbb{R}^2$ - {one point} and $\mathbb{R}^2$ - {2 points} are not homeomorphic. But it is not so easy to prove that. Both are separable metric spaces and they are identical locally - that is, each point has a neighborhood homeomorphic to an open disk. So whatever method we might seek to distinguish the spaces topologically must be "aware of" the entire spaces, not just isolated parts. Furthermore, our intuition that the "number of holes" is just the number of points removed cannot be trusted completely:

If we remove an entire line segment $I^1 = \{ (x,0) \in \mathbb{R}^2 \mid 0 \le x \le 1 \}$, the resulting space is homeomorphic to what we get when we remove just one point.

Also there are different kinds of "holes". The sphere $\mathbf{S}^2$, that is the unit sphere in 3-space ($\{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$) surrounds a "hole"; a standard 2-dimensional torus (see figure below) also surrounds a "hole", but the "holes" are different in some fundamental way.

Spaces with no "holes", what we might call *solid* spaces, are the simplest objects in this world of shapes. These include intervals, the real line, all cubes I^n and all Euclidean spaces $\mathbb{R}^n$.



The key idea in distinguishing the numbers and kinds of "holes" is homotopy: the ability to continuously deform one space to another (e.g. a simpler looking subspace), and the ability to continuously deform one mapping to another. For example, $\mathbb{R}^2$ - {one point} can be continuously deformed to a circle; $\mathbb{R}^2$ - {2 points} cannot. We will say that these two spaces are *homotopically equivalent*, or *have the same homotopy type*.

Every mapping of a circle into $\mathbb{R}^2$ can be continuously deformed to a constant map (i.e. "shrunk to a point"); but there are maps of a circle into $\mathbb{R}^2$ - {one point} that are essential, that cannot be deformed to a constant map. This teaches us that $\mathbb{R}^2$ and $\mathbb{R}^2$ - {one point} are not the same homotopy type, hence are not homeomorphic.

To distinguish $\mathbb{R}^2$ - {one point} from $\mathbb{R}^2$ - {two points}, we get fancier: We can invent a notion of "multiplication" on the set of maps of a circle into a space *X* ; somehow we can take two maps of $\mathbf{S}^1 \to X$ and produce a new map of $\mathbf{S}^1 \to X$ that combines in a meaningful way the original two maps.[♥] Once we have a way to combine maps, we actually can make them into a group. In this sense, the group associated with $\mathbb{R}^2$ - {one point} is cyclic, whereas the group associated with $\mathbb{R}^2$ - {2 points} is not generated by any one element. These are the sorts of ideas involved in the *fundamental group* of a space and its natural companion, *covering spaces*.

♥ For this combining, we don't actually work with maps from a circle into *X;* we work with maps from an interval *[0,1]* where the two endpoints are sent to the same place.

Another basic idea related to "holes" is the notion of one set being the *boundary* of another. A circle in the plane is the boundary of a disk; the 2-dimensional torus (above) in $\mathbb{R}^3$ is the boundary of a solid torus. If our whole world were just the 2-dimensional torus, then we would have circles that do not bound any disks. If you have studied vector calculus, in particular Green's, Stokes's, and Gauss's theorems, then you've seen important situations where the average behavior of a function on a set can be described by its behavior on the *boundary* of the set. (e.g. if **F** is a vector field on $\mathbb{R}^3$, then the integral of the divergence of **F** over some domain is equal to the flux of **F** through the boundary of the domain.) If you were careful about stating those calculus theorems, you know there were subtle issues of orientation: the set and the boundary components had to be oriented consistently. By thinking about things that are "capable of being boundaries" (we call them *cycles*), we are led to develop *homology* theories: A space contains *cycles;* some of the cycles do not bound anything, and those are the ones that capture the holes in the space. We can invent ways to "add" cycles, and again produce groups that describe the shape of the space.

A given space X has homology groups $H_0(X), H_1(X)$, etc., one *group* $H_p(X)$ for each dimension p . Actually, we also will have a lot of groups for each p since there is freedom to choose a *coefficient group*. If the coefficient group is the integers (so we may write $H_p(X;\mathbb{Z})$) then we are assigning abelian groups, i.e. $\mathbb{Z}$-modules; if the coefficient group is a field, say the reals, we write $H_p(X;\mathbb{R})$ and we are choosing a real vector space to measure the "*p-dimensional holes" in X.*

For each way of assigning groups (or vector spaces or other algebra objects) to a space, we also have a way of assigning homomorphisms to continuous maps of spaces. If $f: X \to Y$ is a continuous map of spaces, then we will associate to f a homomorphism of groups $\mathbf{f}_*: H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$. This grand machine takes time to define, and it sometimes is complicated; but the reward is a way to prove that various spaces cannot be topologically equivalent, and that various maps are not homotopic to each other. As a consequence, we obtain famous results such as the Brouwer Fixed Point Theorem and the Jordan Curve Theorem.

There is a way to combine "holes" of some dimensions to produce "holes" of different dimensions. For example, the 2-dimensional torus (shown above) has two "1-dimensional holes", corresponding to two special curves on the surface. When we "multiply" them together, we get a "2-dimensional hole"; this is a rather sophisticated algebraic process analogous to taking the Cartesian product of two sets. On the other hand, a 2-sphere has a "2-dimensional hole" that does not arise from any "1-dimensional holes". In order to make this precise, and gain the ability to distinguish spaces whose individual homology groups are identical, we go to a higher level construction, the **co**homology groups of a space. These support a multiplication between different dimensions and together form a ring. If we have time, the cohomology ring structure, and its applications to orientation of manifolds and duality will be the final topics of our course. Will we get that far??

The first topic in our course is *surfaces*. We want to have a library of spaces that are common in mathematics, reasonably simple, yet topologically varied enough to motivate and illustrate the methods we will develop.

Texts 1. A Basic Course in Algebraic Topology by William Massey (Academic Press, Graduate Texts #127). (Chapters I and V-IX.*)*

2. Algebraic Topology by Allen Hatcher. (Chapters 0, I, Appendix) Available online at http://www.math.cornell.edu/~hatcher/AT/ATpage.html.

Grading: Your grade will be based on weekly homework, two mid-term exams, a final exam, and class participation (which includes regular attendance). The weighting will be approximately:

Midterm Exam #1	20%
Midterm Exam #2	20%
Homework	30%
Final Exam	30%

I expect to use the above weights to compute a numerical average representing the minimum grade you have earned, and then "round up" or add some additional amount if your class participation has been strong; for example, a 3.40 might become an A-, or even A, this way.

Working together: I encourage you to study in groups – it helps a lot when you are trying to learn something if you can explain it to someone else. But your homework is supposed to be your own work. I realize that sometimes it is difficult to draw the line between healthy cooperation and plagiarism, but we all have to be careful about keeping that distinction.

Schedule:

Week of September 25: Midterm Exam #1 7-8:30 p.m. Day and Room to be announced.

Week of October 30: Midterm Exam #2 7-8:30 p.m. Day and Room to be announced.

Wednesday December 13 Final Exam 9:45-11:45 a.m. Room 114 MLH

Note: Monday Sept. 4 is a University holiday. Also we will not have class on Friday Sept 22 and Monday Oct. 2.

Special notes:

This course description represents my current intentions. Changes may be announced in class or by email as needed.

If you wish to contact the Mathematics Department Chair, his office is in 14 MLH. To make an appointment, call 335-0708 or contact the Department Secretary in 14C MLH.

Please let me know if you have a disability, which requires special arrangements.

My own expectation in this course is that we will deal with each other, and with the course material, in a responsible, professional, honorable way, and that we will enjoy working together this term. I welcome your comments, good or bad, about any aspect of the course, any time during the semester, and in the student evaluation forms used at the end.