Lemma 1. Suppose \(X\) is a topological space, \(\sim\) an equivalence relation on \(X\), and \(Q = X/\sim\) the quotient space (i.e. the set of equivalence classes, with the quotient topology). Suppose \(Y\) is a space and \(f : X \to Y\) is a continuous surjective map such that the equivalence classes in \(X\) are precisely the point-inverses under \(f\); that is, for each \(x, y \in X\), \(x \sim y \iff f(x) = f(y)\). If \(X\) is compact and \(Y\) is Hausdorff, then \(Q\) is homeomorphic to \(Y\).

Proof. Let \(\pi : X \to X/\sim\) be the projection map, \(\pi(x) = [x]\). We will be talking about subsets of \(X\) that are unions of points inverses under \(\pi\) (or unions of point inverses under \(f\)). Let us call a subset \(A \subseteq X\) saturated with respect to \(\pi\) if

\[
\forall a \in A, \quad \pi^{-1}\pi(a) \subseteq A.
\]

Similarly, define saturated with respect to \(f\).

If a point inverse intersects a saturated set, then it is contained in that saturated set. Thus the complement of a saturated set must also be saturated (with respect to whatever map is being considered.)

Define a function \(\phi : Y \to Q = X/\sim\) by \(\phi(y) = \pi(f^{-1}(y))\). Note that one of the hypotheses is that equivalence classes in \(X\) under \(\sim\) are precisely the point-inverses under \(f\), i.e. the point inverses under \(f\) are identical to the point inverses under \(\pi\). Since \(f^{-1}(y)\) is exactly \([x]\) for some \(x \in X\), \(\phi(y) = [x]\) is well-defined. Similarly we see that \(\phi\) is a bijection between \(Y\) and \(Q\).

Another consequence of having identical point inverses is that a subset \(A \subseteq X\) is saturated with respect to \(\pi\) if and only if it is saturated with respect to \(f\).

Suppose \(U\) is an open set in \(Q\). Then \(\pi^{-1}(U)\) is a saturated [with respect to \(\pi\)] open set in \(X\). Thus \(K = X - \pi^{-1}(U)\) is closed in \(X\) and also saturated with respect to \(\pi\) and with respect to \(f\). Since \(X\) is compact, \(K\) is compact, so \(f(K)\) is compact in \(Y\). Since \(Y\) is
Hausdorff, \( f(K) \) is closed in \( Y \). Thus \( Y - f(K) \) is open in \( Y \). But \( Y - f(K) = \phi^{-1}(U) \) [check that!]. Thus \( \phi \) is continuous.

We now want to show that \( \phi \) is an open map. This argument also uses the idea of saturated sets. The quotient topology on \( Q \) is defined by saying that a set \( W \) in \( Q \) is open if and only if \( \pi^{-1}(W) \) is open in \( X \). The “only if” part says \( \pi \) is continuous. The “if” part says that [while a quotient map does not have to be an open map] the image of a saturated open set in \( X \) is open in \( Q \).

Let \( V \) be an open set in \( Y \). Then, since \( f \) is continuous, \( f^{-1}(V) \) is open in \( X \). But, furthermore, \( f^{-1}(V) \) is saturated [with respect to \( f \), hence with respect to \( \pi \)]. Thus, by definition of the quotient topology, \( \pi(f^{-1}(V)) \) is open in \( Q \), i.e. \( \phi(V) \) is open in \( Q \).

\[ \square \]