Mayer-Vietoris Sequence

If a space $X$ is the union of two open subsets $A, B$ (or at least the interiors of $A$ and $B$ cover $X$), then there is a long exact sequence relating the homology groups of $X$ to the homology groups of $A, B$ and $A \cap B$. As with the excision theorem, the sequence still is valid if we are dealing with cell-complexes, or some other situation, where $A, B$ are deformation retracts of open sets whose intersection deformation retracts to $A \cap B$.

$$
\begin{align*}
\Delta &\rightarrow H_n(A \cap B) \rightarrow I_* H_n(A) \oplus H_n(B) \rightarrow J_* H_n(A \cup B) \rightarrow \Delta_* H_{n-1}(A \cap B) \\
&\rightarrow \cdots
\end{align*}
$$

To define the homomorphism $\Delta_*$ (and later to prove the sequence exact), we need a lemma based on the idea of “small cubes” (see text for details).

**Lemma 0.1.** (Assuming $X = \text{int } A \cup \text{int } B$) Suppose $w$ is an $n$-chain in $X$. Then there exists $n$-chains $\alpha, \beta$ in $X$ and an $(n+1)$-chain $c$ in $X$ such that

- $\alpha$ is supported in $A$
- $\beta$ is supported in $B$
- $w = \alpha + \beta + \partial w$ as chains

When we say a chain is “supported” in some set, we mean that the chain is a linear combination of singular cubes whose images lie in that set.

The functions $I_*$ is based on the homomorphisms $i_*$ induced by inclusions of $A \cap B$ into $A$ and $B$ respectively.

$$I_*[z] = (A i_*[z], B i_*[z])$$

which we can think of as

$$[z]_{A \cap B} \sim ([z]_A, [z]_B)$$

The function $J_*$ is based on the homomorphisms $j_*$ induced by inclusions of $A$ and $B$ into $A \cup B$, but notice the minus sign.

$$J_* ([\alpha], [\beta]) = A j_* [\alpha] - B j_* [\beta]$$

which we can think of as

$$([\alpha]_A, [\beta]_B) \sim [\alpha - \beta]_{A \cup B}.$$
We now define the function $\Delta_*$. Suppose $z$ is an $n$-cycle in $X$. By Lemma 0.1, we can find chains $\alpha, \beta, c$ such that
\[ z = \alpha + \beta + \partial c , \]
where $\alpha$ is in $A$ (i.e. $\alpha$ is supported in $A$) and $\beta$ is in $B$. For notational convenience, replace $\beta$ with $-\beta$, which, of course, also is supported in $B$. Since $Z$ is a cycle, we know $\partial z = 0$, so we have
\[ z = \alpha - \beta + \partial c , \]
and
\[ 0 = \partial z = \partial \alpha - \partial \beta + \partial \partial c = \partial \alpha - \partial \beta . \]
This says
- $\partial \alpha = \partial \beta$ as chains,
- $\partial \alpha$ is supported in $A$, but also it is supported in $B$, so $\partial \alpha$ is supported in $A \cap B$,
- and $\partial \alpha$ is a cycle in $A \cap B$.

We define
\[ \Delta_*[z]_X = [\partial \alpha]_{A \cap B} . \]

DRAW SOME PICTURES!! to help visualize the function $\Delta_*$. For example, if $X$ is the $n$-sphere expressed as the union of $A=$upper hemisphere, and $B=$lower hemisphere, then all of $X$ is an $n$-cycle, and this cycle is mapped by $\Delta_*$ to the $(n-1)$ dimensional homology class represented by the equator $(n-1)$-sphere.

**Lemma 0.2 (Unassigned HW).** Show the homomorphisms (in particular $\Delta_*$) are well-defined.

We next consider how to prove that the sequence is exact. We work out one part below, another part is assigned HW, and the third part is “unassigned HW”.

**Proof that the sequence is exact at $H_n(A \cup B)$**. We have
\[ I_* : H_n(A) \oplus H_n(B) \xrightarrow{J_*} H_n(A \cup B) \xrightarrow{\Delta_*} H_{n-1}(A \cap B) \rightarrow . \]
First,
\[ ([\alpha]_A, [\beta]_B) \sim [\alpha - \beta]_{A \cup B} \sim [\partial \alpha]_{A \cap B} . \]
But $\alpha$ is a cycle, so its boundary $= 0$. 

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The converse takes a little more work. Suppose \( z \) is a cycle in \( X \) and \( \Delta_*[z] = 0 \in H_{n-1}(A \cap B) \). This means that when we write
\[
z = \alpha - \beta + \partial c,
\]
there is an \( n \)-chain \( w \) in \( A \cap B \) such that \( \partial \alpha = \partial w \), i.e. \( \partial(\alpha - w) = 0 \).

Since \( w \) is a chain in \( A \cap B \), it is, in particular, supported in \( A \). Thus \( \alpha - w \) is a cycle in \( A \). Similarly, \( \beta - w \) is a cycle in \( B \). But then we have
\[
( [\alpha - w]_A, [\beta - w]_B ) \xrightarrow{J_*} [\alpha - w - (\beta - w)]_{A \cup B},
\]
which says \( [z - \partial c] \in \text{image } J_* \). We are almost done ... We want \( [z] \in \text{image } J_* \). But since \( z \) and \( z - \partial c \) differ just by a boundary, they define the same homology class, i.e.
\[
[z] = [z - \partial c] \in \text{image } J_*.
\]