Practice Exercises
on
Computing $\pi_1(X)$

Our tools for computing $\pi_1(X)$:

- The group $\pi_1$ is invariant under homotopy type. In particular, if $A \subseteq X$ and $X$ strongly deformation-retracts to $A$, then the inclusion-induced homomorphism $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism.
- The circle has $\pi_1(S^1) \cong \mathbb{Z} \cong \langle a \rangle$. Moreover, for each integer $n$, the loop $[0, 1] \to S^1$ given by $\alpha(t) = [\cos(2\pi nt), \sin(2\pi nt)]$ represents the element $\alpha^n$.

**Exercise 1.** Outline proof that $\pi_1(S^1) \cong \mathbb{Z}$, as presented in class handout.

- The 2-sphere has $\pi_1(S^2) = \{1\}$. (similarly for any higher dimensional sphere.)
- For product spaces, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.
- The Seifert-Van Kampen theorem: Suppose $X = A \cup B$, where $A$ and $B$ are open in $X$ and $A \cap B$ is path-connected. Suppose also we have presentations

  $\pi_1(A) \cong < a_1, a_2, \ldots | R_1, R_2, \ldots >$
  $\pi_1(B) \cong < b_1, b_2, \ldots | S_1, S_2, \ldots >$
  $\pi_1(A \cap B) \cong < c_1, c_2, \ldots | T_1, T_2, \ldots >$.

  Then $\pi_1(X) \cong < a_1, a_2, \ldots, b_1, b_2, \ldots | R_1, R_2, \ldots, S_1, S_2, \ldots, (i_A)_*(c_1) = (i_B)_*(c_1), (i_A)_*(c_2) = (i_B)_*(c_2), \ldots >$

**Exercise 2.** Prove that $\pi_1(X)$ has a generating set consisting of [homotopy classes of] loops that are contained in $A$ and loops contained in $B$. 
Exercise 3. $\pi_1(S^1 \vee S^1) \cong$

Exercise 4. $\pi_1(S^1 \vee S^1 \ldots S^1) \cong$

Exercise 5. $\pi_1(S^2 \vee S^2) \cong$

Exercise 6. $\pi_1(S^2 \vee S^2 \ldots S^2) \cong$

Exercise 7. $X = \text{finite graph}$.

Definition. The genus of a connected graph $G$ is a number $n$, calculated as follows. Let $T$ be a maximal tree in $G$ and let $\{e_1, \ldots, e_n\}$ be the edges of $G$ that are not contained in $T$. The number of such edges is the genus of $G$. Note that because that number $n$ is the rank of $\pi_1(G)$, the genus is well-defined, independent of which tree we use to calculate it.

Theorem. The Euler characteristic of a finite graph is a topological invariant, in fact it is a homotopy-type invariant.

Exercise 8. Prove this theorem.

Exercise 9. $X = \text{disk } D^2 \text{ with 3 interior points removed}$.

Definition. Consider a standard picture of a surface $F^2 = T_1^2 \# T_2^2 \# \ldots \# T_n^2$ in $\mathbb{R}^3$. (For $n = 1$, this is a torus; for $n = 2$, this is a double-torus; etc.) The surface $F^2$ bounds a 3-manifold-with-boundary called a handlebody of genus $n$. (So a handlebody of genus 1 is a solid torus; a handlebody of genus 2 is a solid double torus; etc.)

Exercise 10. $X = \text{handlebody}$

Hint. There are (at least) two ways to approach the previous exercise. One is to find a 1-dimensional graph inside the handlebody that is a SDR of the handlebody. Another approach is to construct handlebodies inductively, viewing a handlebody of genus $n$ as the union of a solid torus and a handlebody of genus $n - 1$. 

©J. Simon, all rights reserved
**Theorem.** (This theorem says that if we “puncture” a 3-manifold, we do not change the fundamental group. We can define “puncture” as removing a point, or as removing a solid open 3-ball. Since the proof in the case of removing a point depends on the case of removing a small 3-ball, we will state the more basic version of the theorem. Do not be frightened by all the explanation - this is a pretty simple idea.)

Suppose $M$ is a 3-manifold (with or without boundary). Let $B_0$ be a closed 3-ball contained in the interior of $M$. Suppose, further, that the sphere $S = \text{bdy}(B_0)$ is “bi-collared” in $M$, which means there exists an open neighborhood $U$ of $S$ in $M$ homeomorphic to $S \times (-1,1)$, where $S \times (-1,0] = U \cap B_0$ and $S \times [0,1) = U \cap (M - \text{int}(B_0))$. Let $W$ be the 3-manifold with boundary obtained by removing $\text{int}(B_0)$ from $M$. Then $\pi_1(W) \cong \pi_1(M)$.

**Exercise 11.** Prove this theorem.
(Hint: Let $A = W \cup S \times (-1,0]$ and let $B = B_0 \cup [0,1)$ and see what Van Kampen’s theorem tells you. You may need to do some deformation-retracting along the way.)

**Exercise 12.** $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

**Theorem.** We can apply Van Kampen’s theorem to situations in which the “pieces” are deformation retracts of open sets; the do not need to be open themselves. In particular, this works if $X$ is a CW-complex, $X = A \cup B$, where $A, B$, and $A \cap B$ are all subcomplexes.

**Exercise 13.** $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$.

**Theorem.** If $X$ is a CW-complex, then $\pi_1(X)$ is generated by loops in $X^{(1)}$ and all the relations in $\pi_1(X)$ are “visible” in $X^{(2)}$. In particular, the inclusion-induced homomorphism $i_* : \pi_1(X^{(2)}) \to \pi_1(X)$ is an isomorphism.