Note: Assume all spaces are metric, or whatever else is nice enough to make the theorems work. We don’t want to worry too much about pathological spaces. Also all maps are continuous.

**Problem 1**
Suppose $W$ is a cell complex consisting of the union of two sub-complexes that intersect in a subcomplex. In other words, $X$ and $Y$ are cell-complexes, $W = X \cup Y$, and $X \cap Y$ is a sub-complex of each of $X$ and $Y$.

(a) State (no proof) the theorem that expresses the Euler characteristic $\chi(W)$ in terms of the characteristics of $X$, $Y$, and $X \cap Y$.

(b) Use your result in part (a) to calculate $\chi(X \vee Y)$.

(c) Suppose $X$ and $Y$ are compact connected 2-manifolds $X \# Y$. Use your result from part (a) to calculate $\chi(X \# Y)$.

**Problem 2**
If $Y$ is a solid space and $X$ is any space, then each map $f : X \rightarrow Y$ is homotopic to a constant map.

**Problem 3**
Write explicit homotopies to show that loop-homotopy (i.e. $\simeq_{x_0}$) is an equivalence relation.

**Problem 4**
Prove that multiplication of loop classes is well defined, i.e. if $f$ and $g$ are loops in $X$ at $x_0$, then $[f][g] = [fg]$ is a well-defined operation on loop classes.

**Problem 5** Outline a proof that $X$ path connected $\implies$ for each $x_0, y_0 \in X$, the groups $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$ are isomorphic.
Problem 6
Suppose $a, b, c,$ are points in a space $X$, $f$ is a path from $a$ to $b$, $g$ is a path from $b$ to $c$, $\hat{f}$ is another path from $a$ to $b$, and $\hat{g}$ is another path from $b$ to $c$ such that $f \simeq_{a,b} \hat{f}$ and $g \simeq_{b,c} \hat{g}$. Prove $fg \simeq_{a,c} \hat{f} \hat{g}$.

Problem 7
If $f : X \to Y$ is a map,
(a) Define the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.
(b) Prove $f_*$ is well-defined.

Problem 8
State and prove the “functorial” properties for map-induced homomorphisms of fundamental groups.

Problem 9
Prove: If $f, g : X \to Y$ are maps, $x_0 \in X$, and $f \simeq_{x_0} g$, then $f_* = g_*$ as homomorphisms from $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

Problem 10
Outline a proof that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Problem 11
Prove that none of the three spaces $S^2 \times S^1$, $S^3$, $S^1 \times S^1 \times S^1$ is homeomorphic to any of the others (These are several compact three-manifolds, and you are showing they are topologically different from each other). State clearly any theorems you use.

Problem 12
Show there is no retraction of $D^2$ to its boundary.

Problem 13
Suppose $T$ is a 2-dimensional torus in $\mathbb{R}^3$. Assume that the origin $0 \notin T$.

Prove that $\mathbb{R}^3 - \{0\}$ does not retract to $T$. 