22M:201
Fall 05
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Sample Problems
for
Final Exam

NOTE: The instruction, "Prove ..." is ambiguous. Every proof is a mixture of formalism and rhetoric, unless perhaps one is writing exercises in symbolic logic. Your goal here should be to write proofs that are **less** than 2 pages long (perhaps even just one well-designed page) that show that you really understand what issues are important for that particular theorem. For example, when a person is first defining homology groups, the issue of degenerate cubes needs to be addressed. But by the time you are worrying about why the Meyer-Vietoris sequence is exact, it seems [to me anyway] better to treat a "chain" as a linear combination of cubes, rather than as a coset.

Problem 1. *Suppose* $X = \{a, b\}$ *is a space consisting of exactly two points. Compute the homology groups* $H_n(X)$ *directly from definitions.*

Problem 2. *Suppose a space* X *consists of exactly two path-components* A, B. *Prove that for all* n,
$H_n(X) \cong H_n(A) \oplus H_n(B)$.

Problem 3. *Suppose* $A \subseteq X$ *has the property that for each path component* W *of* X, $A \cap W \neq \emptyset$. *Prove* $H_0(X, A) = 0$.

Problem 4. *Calculate the homology groups of the space* $X = S^1 \vee S^1$ *by using the Mayer-Vietoris sequence.*

Problem 5. *Calculate the homology groups of the space* $X = S^1 \vee S^1$ *by using the long exact sequence of a pair (and not using the M-V sequence).*

Remark: Let A *and* B *denote the two copies of* S^1 *that make up* X. *Analyze the sequence for the pair* (X, B).

Problem 6. *Suppose* $p \in X$ *and let* $A = \{p\}$. *Use the long exact sequence for the pair* (X, A) *to show that* $H_n(X, A) \cong \widetilde{H}_n(X)$.

Problem 7. *[Actually several possible problems]*

For the long exact sequence of a pair (X, A), define the various homomorphisms, and prove the sequence is exact.
Remark: Aim for a proof that is a satisfying mixture of formalism and intuition.

Problem 8. *[Actually several possible problems]*
For the Meyer-Vietoris sequence of a triple (X, A, B), where $X = A \cup B$, define the various homomorphisms and prove the sequence is exact.
Remark: Aim for a proof that is a satisfying mixture of formalism and intuition.

Problem 9. *Calculate the homology groups $H_k(S^n)$ using the Mayer-Vietoris sequence.*

Problem 10. *Calculate the homology groups $H_k(S^n)$ by using the long exact sequence of a pair (and not using the M-V sequence).*

Problem 11. *State the "functorial properties" of homology theory. Prove that homology groups are topological invariants.*

Problem 12. *Suppose A is a retract of X. Prove that for each n, $H_n(A)$ is isomorphic to a subgroup of $H_n(X)$, and also $H_n(A)$ is a homomorphic image of $H_n(X)$.*

Problem 13. *Suppose $f, g : X \to Y$ are maps. Prove that if f is homotopic to g, then for each n, $f_*^n = g_*^n : H_n(X) \to H_n(Y)$.*

Problem 14. *Define what it means for two spaces to be of the same* homotopy type*. Prove that homology groups are homotopy-type invariants. State whatever lemma(s) you use for this, but do not prove the lemma(s).*

Problem 15. *Define the* Euler characteristic *of a finite cell-complex. Prove (i.e. outline a proof) that the Euler characteristic of a cell complex can be computed in terms of the homology groups of the underlying topological space (and so conclude that Euler characteristic is a topological invariant, in fact a homotopy-type invariant).*

Problem 16. *[several problems]*
Calculate the homology groups of $T^2 = S^1 \times S^1$, $T^2 \# T^2$, $P^2 = D^2 \cup M^2$, and $K^2 = M^2 \cup M^2$ using the Meyer-Vietoris sequence.