Sample Problems
for
Final Exam

NOTE: The instruction, “Prove . . . ” is ambiguous. Every proof is a mixture of formalism and rhetoric, unless perhaps one is writing exercises in symbolic logic. Your goal here should be to write proofs that are **less** than 2 pages long (perhaps even just one well-designed page) that show that you really understand what issues are important for that particular theorem. For example, when a person is first defining homology groups, the issue of degenerate cubes needs to be addressed. But by the time you are worrying about why the Meyer-Vietoris sequence is exact, it seems [to me anyway] better to treat a “chain” as a linear combination of cubes, rather than as a coset.

**Problem 1.** Suppose $X = \{a, b\}$ is a space consisting of exactly two points. Compute the homology groups $H_n(X)$ directly from definitions.

**Problem 2.** Suppose a space $X$ consists of exactly two path-components $A, B$. Prove that for all $n$, $H_n(X) \cong H_n(A) \oplus H_n(B)$.

**Problem 3.** Suppose $A \subseteq X$ has the property that for each path component $W$ of $X$, $A \cap W \neq \emptyset$. Prove $H_0(X, A) = 0$.

**Problem 4.** Calculate the homology groups of the space $X = S^1 \vee S^1$ by using the Mayer-Vietoris sequence.

**Problem 5.** Calculate the homology groups of the space $X = S^1 \vee S^1$ by using the long exact sequence of a pair (and not using the M-V sequence).

*Remark:* Let $A$ and $B$ denote the two copies of $S^1$ that make up $X$. Analyze the sequence for the pair $(X, B)$.

**Problem 6.** Suppose $p \in X$ and let $A = \{p\}$. Use the long exact sequence for the pair $(X, A)$ to show that $H_n(X, A) \cong \tilde{H}_n(X)$.

**Problem 7.** [Actually several possible problems]
For the long exact sequence of a pair \((X, A)\), define the various homomorphisms, and prove the sequence is exact.

Remark: Aim for a proof that is a satisfying mixture of formalism and intuition.

**Problem 8.** [Actually several possible problems]

For the Meyer-Vietoris sequence of a triple \((X, A, B)\), where \(X = A \cup B\), define the various homomorphisms and prove the sequence is exact.

Remark: Aim for a proof that is a satisfying mixture of formalism and intuition.

**Problem 9.** Calculate the homology groups \(H_k(S^n)\) using the Mayer-Vietoris sequence.

**Problem 10.** Calculate the homology groups \(H_k(S^n)\) by using the long exact sequence of a pair (and not using the M-V sequence).

**Problem 11.** State the “functorial properties” of homology theory. Prove that homology groups are topological invariants.

**Problem 12.** Suppose \(A\) is a retract of \(X\). Prove that for each \(n\), \(H_n(A)\) is isomorphic to a subgroup of \(H_n(X)\), and also \(H_n(A)\) is a homomorphic image of \(H_n(X)\).

**Problem 13.** Suppose \(f, g : X \to Y\) are maps. Prove that if \(f\) is homotopic to \(g\), then for each \(n\), \(f_\ast^n = g_\ast^n : H_n(X) \to H_n(Y)\).

**Problem 14.** Define what it means for two spaces to be of the same homotopy type. Prove that homology groups are homotopy-type invariants. State whatever lemma(s) you use for this, but do not prove the lemma(s).

**Problem 15.** Define the Euler characteristic of a finite cell-complex. Prove (i.e. outline a proof) that the Euler characteristic of a cell complex can be computed in terms of the homology groups of the underlying topological space (and so conclude that Euler characteristic is a topological invariant, in fact a homotopy-type invariant).

**Problem 16.** [several problems]

Calculate the homology groups of \(T^2 = S^1 \times S^1\), \(T^2 \# T^2\), \(P^2 = D^2 \cup M^2\), and \(K^2 = M^2 \cup M^2\) using the Meyer-Vietoris sequence.