1. Making Wishes

We have seen how the idea of “strong deformation retraction” can be generalized to the notion of two spaces being homotopy equivalent, i.e. of the same homotopy type. In particular,

**Definition 1.** A space $X$ is contractible if $X$ has the homotopy type of a point.

Notice in the above definition, we are not saying that $X$ deformation retracts to any particular point of $X$. We would welcome a theorem saying that if a space $X$ is contractible, then $X$ does deformation retract to one of its points. Perhaps we could even hope for a theorem saying that $X$ can be strong deformation retracted to any point we wish.

**Wish 1.** If $X$ is contractible, then $\exists x \in X$ such that $X$ strong deformation retracts to $X$. Perhaps we might even hope for: $\forall x \in X$, $X$ s-d-r’s to $X$.

**Example 1.** Let $X_1$ be the infinite “fan” of arcs consisting of all the line segments in $\mathbb{R}^2$ from the origin to the points $(1, \frac{1}{n})$, $n = 1, 2, 3, \ldots$. Let $X$ be the closure of $X_1$, that is adjoin to $X_1$ the segment from the origin to the point $(1,0)$. The we can (strong) deformation retract $X$ to the origin, but we cannot deform it to the point $p = (\frac{1}{2},0)$ without moving $p$.

It is possible to build a more complicated space (see Hatcher Exercises) that is contractible but does not deformation retract to any of its points.

[Note on an English language subtlety in use of the word any: If we say, “This space contracts to any of its points”, that means $\forall x \in X$, $X$ contracts to $x$. If we say, “This space does not contract to any of its points”, that means $\not\exists x \in X$ such that $X$ contracts to $x$. There is...]

©ALL RIGHTS RESERVED
no quick one-word way to say, “This space contracts to some points but not to others.”]

Here is another example of a theorem that is too good to be true:

**Wish 2.** Suppose $A$ is a contractible subset of a space $X$. Then the quotient space of $X$ modulo $A$ is of the same homotopy type as $X$, that is $X \simeq X/A$.

To see that this is too much to hope for, we first prove:

**Lemma 0.1.** If $X, Y$ are spaces with $X$ path-connected, and $X \simeq Y$, then $Y$ is path-connected.

The next example is a space $X$ that is not path-connected, with a contractible subset $A$ such that $X/A$ is path-connected. Thus $X$ cannot be homotopy equivalent to $X/S$.

**Example 2.** Let $X$ be the closure of the graph of the function $y = \sin(1/x)$. That is, $X = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y = \sin(1/x)\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$.

Since wishes are free, here are more:

**Wish 3.** If $A \subseteq X$, and the inclusion map $i : A \to X$ is a homotopy equivalence, then $X$ strong deformation retracts to $X$.

**Wish 4.** If $X = A \cup B$ where $A$, $B$, and $A \cap B$ are all contractible, then $X$ is contractible.

**Wish 5.** Suppose $X, Y$ are spaces, $A \subseteq X$, $f : A \to Y$ a continuous function, and $W_f = X \cup_f Y$, that is $W_f$ is the space obtained by attaching $X$ to $Y$ via $f$. Suppose we have another map $g : A \to Y$ that is homotopic to $f$, and let $W_g$ be the space obtained by attaching $X$ to $Y$ via $g$. Then $W_f \simeq W_g$.

**Wish 6.** Every space $X$ is Hausdorff, even normal. (So theorems like Urysohn's Lemma and the Tietze Extension Theorem can be used.)

**Wish 7.** Each point $x \in X$ has an open neighborhood $U(x)$ such that $U$ strong deformation retracts to $x$. (Note this would imply that $X$ is locally path connected.)

**Wish 8.** Each closed subset $A \subseteq X$ has an open neighborhood $U(A)$ such that $U$ strong deformation retracts to $A$. In fact, if $X = A \cup B$, where $A$ and $B$ are closed, then there are neighborhoods $U(A)$, $V(B)$ such that $U$ sdr’s to $A$, $V$ sdr’s to $B$, and $U \cap V$ sdr’s to $A \cap B.$
Remark on the last “wish”: One of our goals is to prove theorems that let us understand the homology groups and fundamental group of a space by seeing the space as being built from simple pieces (e.g. cells). Typically, the theorems require that the pieces be open sets. But, just as typically, we do the actual building with compact sets (such as cells). The last “wish” would extend the open-set versions to the compact-set constructions.

2. Wishes Can Come True

In the above list of wishes, assume that each space being discussed is a CW-complex, and each subset discussed is a sub-complex. Then all your wishes come true!

Here is just one sample of the kinds of calculations that become possible when we restrict our attention to spaces that are “nice”.

Example 3. Let $X = S^2 \vee S^1$. Let $A$ be an arc. Let $W$ be obtained by attaching $A$ to $X$ at the endpoints of $A$. Then all seven [check that] topologically different spaces are homotopy equivalent.