0.1. **Don’t worry yet about “degenerate” cubes.** I suggest you study this section in the order: first = class notes, second = this handout, third = read the text and do the homework problems.

0.2. **Just to make sure you understand how “faces” are defined.** Suppose $T$ is a singular 3-cube in $X$, that is $T : I^3 \to X$. Then $T$ has six “faces”. Each face of $T$ is a singular 2-cube, that is a map from $I^2 \to X$. The idea is easier than the notation we eventually end up using.

There are 3 directions in $\mathbb{R}^3$. In each direction, $T$ has a “far” face and a “near” face: the text calls these the “back” and “front” faces respectively. We have to specify six functions from $I^2$ into $X$. So for each of the six functions, we have to decide where to send the points $(a,b) \in I^2$. Keep saying it over and over again: “Each face is a function. A given face is the function that sends points $(a,b)$ to ...”:

<table>
<thead>
<tr>
<th>Direction</th>
<th>front face</th>
<th>back face</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$-direction</td>
<td>$T(0,a,b)$</td>
<td>$T(1,a,b)$</td>
</tr>
<tr>
<td>$y$-direction</td>
<td>$T(a,0,b)$</td>
<td>$T(a,1,b)$</td>
</tr>
<tr>
<td>$z$-direction</td>
<td>$T(a,b,0)$</td>
<td>$T(a,b,1)$</td>
</tr>
</tbody>
</table>

0.3. **Define the “boundary”.** The boundary of an $n$-cube $T$ is a linear combination of $(n-1)$-cubes. The boundary of the boundary of $T$ is then a linear combination of $(n-2)$ cubes. We want that boundary to be 0. The “boundary” is actually a homomorphism from the group of $n$-cubes to the group of $(n-1)$-cubes, denoted $\partial_n : Q_n(X) \to Q_{n-1}(X)$. The claim is that the homomorphism $\partial_{n-1} \circ \partial_n : Q_n(X) \to Q_{n-2}(X)$ is the zero homomorphism.

Consider the case where $T$ is a 2-cube. Then $\partial_2(T)$ is a linear combination of 1-cubes, and the boundary of that is a linear combination of 0-cubes, essentially a linear combination of points of $X$. Each point in this combination appears twice, and we need to make sure that it appears once with a $(+)$ and once with a $(-)$. That is accomplished by the identities $A_iA_j(T) = A_{j-1}A_i(T)$ etc. displayed in text (7.2.1).

Here is an example: $T$ is a 3-cube. What is $(A_1A_3T)(s)$? (Note: don’t think of this as functional composition that is associative; $A_i$ of
\((A_3T)(s)\) would be \(A_1\) of a point in \(X\), which doesn’t make sense. Also \(A_3T(s)\) would not make sense, since the domain of \(A_3T\) is \(I^2\), not \(I^1\).)

\(A_3T\) is a 2–cube. \(A_1(A_3T)\) is the front face of that 2–cube in the first coordinate direction. So \(A_1(A_3T)(s) = (A_3T)(0, s)\). Now, what does \((A_3T)\) do to a pair \((t, s)\)? It takes the pair to a point on the front face of \(I^3\) in the third direction, and then applies \(T\). So \((A_3T)(0, s) = T(0, s, 0)\). Similarly, \(A_1B_3T(s) = T(0, s, 1)\).

**HOMEWORK:** Prove the fourth identity in (7.2.1), that is, prove that the functions \(B_iA_j(T)\) and \(A_j−1B_i(T)\) are identical.

(Hint: Both of these are \((n−2)\)–cubes. So start with a point \((a_1, \ldots, a_{n−2})\) and see where it is sent under each of the two maps.)

0.3.1. **NOW we define the boundary homomorphism.** The function

\[ \partial_n : C_n(X) \to C_{n−1}(X) \]

is given by its action on each basis element, that is its action on each \(n\)–cube:

\[ \partial_n(T) = \sum_{i=1}^{n} (-1)^i ((A_iT) - (B_iT)) . \]

**HOMEWORK:** Prove that \((\partial \partial) = 0\). Since this will be true if and only if it is true for generators of \(Q_n(X)\), it is necessary and sufficient for you to prove that for each singular \(n\)–cube \(T\),

\[ \partial_{n−1}(\partial_n(T)) = 0 \in C_{n−2}(X) . \]

(Hint: The proof follows directly from the identities (7.2.1); you just need to do the careful bookkeeping.)

0.4. **Provisional definition of cycles, bounds, and homology groups.** We are going to define the *cycles*, \(Z_n(X)\) to be those combinations of \(n\)–cubes that have 0 boundary. That is,

\[ Z_n(X) = \ker \partial_n . \]

These \(n\)–cycles are trying to record the existence of \(n\)–dimensional “holes” in \(X\). But if a cycle is actually the boundary of something, then there’s no “hole” to record. With this in mind, we define a
boundary to be any combination of \( n \)–cubes that is itself the boundary of some combination of \( (n + 1) \)–cubes. That is,

\[
B_n(X) = \text{image } \partial_{n+1}.
\]

We then define the homology groups of \( X \) by

\[
H_n(X) = \frac{Z_n(X)}{B_n(X)}.
\]

0.5. **But this give rise to too many cycles.** If we just proceed as above, we have too many cycles. For example, if the 1–cube \( f : [0, 1] \to X \) is a constant path, \( f(s) = x_0 \) for all \( s \in [0, 1] \), then \( f \) is a 1–cycle, that is \( f \in Z_1(X) \), since the front and back 0–faces of \( f \) are identical, so their difference is 0. On the other hand, \( f \notin B_1(X) \): the boundary of any 2–cube is a linear combination of four 1–cubes, so each element of \( B_1(X) \) is a combination of an even number of 1–cubes. A 1–cube that actually is a constant function is called a degenerate 1–cube.

Similarly, suppose \( f : [0, 1] \to X \) is a loop, i.e. path with \( f(0) = f(1) \). Consider the degenerate 2–cube \( T \), given by \( T(s, t) = f(t) \). Then

\[
\begin{align*}
A_1T(x) &= T(0, x) = f(x) & B_1T(x) &= T(1, x) = f(x) \\
A_2T(x) &= T(x, 0) = f(0) & B_2T(x) &= T(x, 1) = f(1)
\end{align*}
\]

So \( \partial_2T = 0 \), i.e. \( T \in Z_2(X) \); but \( T \) is not a boundary. The moral is that if we include degenerate \( n \)–cubes, then \( H_n(X) \) will be bigger than we want if our goal is to keep track of “holes” to describe the shape of \( X \). The solution is to get rid of these degenerate cubes from the beginning.

0.6. **Define degenerate \( n \)–cubes.** A singular cube \( T : I^n \to X \) is called degenerate if there is some index \( i \) such that the map \( T \) factors as \( (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \to (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \to X \).

In the book’s words, \( T \) is independent of the \( i^{th} \) coordinate.

Define \( D_n(X) = \text{subgroup of } Q_n(X) \) generated by all degenerate \( n \)–cubes. Now define

\[
C_n(X) = \frac{Q_n(X)}{D_n(X)}.
\]

After checking that the boundary homomorphism respects degeneracy (that is, the function \( \partial_n : Q_n \to Q_{n-1} \) takes the subgroup \( D_n \) into \( D_{n-1} \), we can define the \{adjusted, modified, improved, corrected\} version of \( \partial_n \) as a map of \( C_n(X) \to C_{n-1}(X) \). Then define \( Z_n \subseteq C_n \) and \( B_n \subseteq C_n \) and \( H_n(X) = Z_n(X)/B_n(X) \).

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0.7. **Homology groups of a one-point set** $X$. In dimensions $n \geq 1$, all cubes are degenerate. So

$$C_n(X) = \{0\} \implies Z_n(X) = \{0\} \implies H_n(X) = \{0\}.$$ 

That leaves dimension $n = 0$. There is only one 0–cube, so $Q_0(X) \cong \mathbb{Z}$. There are no “degenerate” 0–cubes, so $C_0 = Q_0 / \{0\} \cong \mathbb{Z}$. All 1–cubes in $X$ are degenerate, so $C_1 = Q_1 / D_1 = \{0\}$. Thus $B_0 = \partial_1(C_1) = \{0\}$. So $H_0(X)$ will equal $Z_0(X)$, once we decide which 0–chains to call cycles. If you go back through the definitions, you will realize that we never defined the boundary of a 0–chain, so we don’t yet know which of those we will say have 0 boundary. That is for the next [sub-]section VII.2.2.