GROUP PRESENTATIONS AND TIEZTE TRANSFORMATIONS
(continued)

Here is a step-by-step proof, using Tietze transformation, that the presentations

\[ \langle a, b \mid a^2, b^3 \rangle \]

and

\[ \langle x, y \mid xyx = yxy \rangle \]

present isomorphic groups.

Start with \( \langle x, y \mid xyx = yxy \rangle \), and introduce new generating symbols as “nicknames” for existing elements.

\[ \langle x, y \mid xyx = yxy \rangle \] (1)
\[ \cong \langle x, y, a \mid xyx = yxy, a = xyx \rangle \] (2)
\[ \cong \langle x, y, a, b \mid xyx = yxy, a = xyx, b = xy \rangle \] (3)

Add new relation that is a consequence of the ones already there. Since \( a^2 = xyxxyx \), and \( b^3 = xyxyxy = (xyx)(yxy) = (xyx)(xyx) \), our group now is seem to be isomorphic to

\[ \langle x, y, a, b \mid xyx = yxy, a = xyx, b = xy, a^2 = b^3 \rangle \] . (4)

But now, the relation \( xyx = yxy \) can be deduced from the other three, that is, in any group with elements \( a, b, x, y \), the three equations \( a = xyx \), \( b = xy \), and \( a^2 = b^3 \) together imply that \( xyx = yxy \). So in the presence of the other three relations, the first one is redundant and can be deleted without changing the group. More formally, in the free group of rank 4 generated by \( \{a, b, x, y\} \), the normal subgroup

\[ \langle \langle xyxy^{-1}x^{-1}y^{-1}, a^{-1}xyx, b^{-1}xy, a^2b^{-3} \rangle \rangle \] (5)
\[ = \langle \langle a^{-1}xyx, b^{-1}xy, a^2b^{-3} \rangle \rangle . \] (6)

The first two relations, which remind us how we introduced “a” and “b” as words in \( x \) and \( y \), can also be used to express \( x \) and \( y \) in terms of \( a \) and \( b \). We just solve the two equations for unknowns \( x \) and \( y \).
The second equation says \( y = x^{-1}b \). Substitute that into the first equation to get \( a^{-1}x(x^{-1}b)x = 1 \), i.e. \( a^{-1}bx = 1 \), which implies \( x = b^{-1}a \). Now substitute that back into the equation \( y = x^{-1}b \) to get \( y = (b^{-1}a)^{-1}b = a^{-1}b^2 \). Since the equations giving \( x \) and \( y \) in terms of \( a \) and \( b \) are consequences of the other equations, our group is now isomorphic to

\[
\langle a, b, x, y \mid a^{-1}xyx, b^{-1}xy, a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2 \rangle.
\]

But now you can check that the last two equations imply the first two (even in the free group on \( \{a, b, x, y\} \) symbols): Just replace each \( x \) and each \( y \) by \( b^{-1}a \) and \( a^{-1}b^2 \) respectively to show \( a^{-1}xyx = 1 \) and \( b^{-1}xy = 1 \). Thus we have

\[
\langle a, b, x, y \mid a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2 \rangle.
\]

Finally, we eliminate each generator \( x, y \) and the single relation that defines it as some word in the other generators. This leaves just

\[
\langle a, b, x, y \mid a^2b^{-3} \rangle.
\]

Another way to prove

\[
\langle x, y | xyx = yxy \rangle \cong \langle a, b | a^2 = b^3 \rangle
\]

would be to exhibit isomorphisms. Define \( \phi : \langle x, y | xyx = yxy \rangle \rightarrow \langle a, b | a^2 = b^3 \rangle \) by \( \phi(x) = b^{-1}a \) and \( \phi(y) = a^{-1}b^2 \). Define \( \psi : \langle a, b | a^2 = b^3 \rangle \rightarrow \langle x, y | xyx = yxy \rangle \) by \( \psi(a) = xyx \) and \( \psi(b) = xy \). Then show that the functions \( \phi \) and \( \psi \) define homomorphisms and that they are inverses of each other.