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Prove or give a counterexample:

Suppose a family of sets $\{A_\alpha\}_{\alpha \in J}$ is linearly ordered by set inclusion. If the index set J is uncountable, then the set $\bigcup_{\alpha \in J} A_\alpha$ is uncountable.

Counterexample: Let $A_r = \{q \in \mathbb{Q}$ such that $q \leq r\}$, and consider the family of sets $\{A_r\}_{r \in \mathbb{R}}$. Suppose that $r_1 \neq r_2 \in \mathbb{R}$ and without loss of generality, assume $r_1 < r_2$. Then for any $q \in A_{r_1}$, we have $q \leq r_1 < r_2$, so $q \in A_{r_2}$ and $A_{r_1} \subset A_{r_2}$. Further, there must exist a rational number q_0 such that $r_1 < q_0 < r_2$, so $q_0 \in A_{r_2}$ but $q_0 \notin A_{r_1}$, and thus $A_{r_1} \subsetneq A_{r_2}$. Finally, note that the index set $\mathbb{R}$ is uncountable, but $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{Q}$, which is a countable set.