22M:132
Fall 07
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Urysohn's Lemma

(Relates to text Sec. 32-36)

Introduction. Saying that a space X is *normal* turns out to be a very strong assumption. In particular, normal spaces admit a lot of continuous functions:

Theorem (Urysohn's Lemma)**.** *If A, B are disjoint closed sets in a normal space X, then there exists a continuous function $f : X \to [0, 1]$ such that $\forall a \in A$, $f(a) = 0$ and $\forall b \in B$, $f(b) = 1$.*

This lemma has several "big" applications:

(1) Urysohn Metrization Theorem (Section 34) *If X is a normal space with a countable basis (i.e. second-countable), then we can use the abundance of continuous functions from X to $[0, 1]$ to assign numerical coordinates to the points of X and obtain an embedding of X into $\mathbb{R}^\omega$. From this we see that every second-countable normal space is a metric space.*

(2) Tietze Extension Theorem (Section 35) (This theorem is a useful technical tool, rather than a big conceptual climax like the Metrization theorem.) *Suppose A is a subset of a space X and $f : A \to [0, 1]$ is a continuous function. If X is normal and A is closed in X, then we can find a continuous function from X to $[0, 1]$ that is an extension of f.*

(3) Embedding manifolds in $\mathbb{R}^N$ (Section 36) A space X is called a *topological n-manifold* if each point $x \in X$ has an open neighborhood $U(x)$ such that U is homeomorphic to an open n-ball.

Usually people insist that X be Hausdorff, sometimes 2nd-countable, sometimes "paracompact" [defined later in the text]. Right now our text assumes X is Hausdorff and has a countable basis, so we will assume that as part of the definition of "manifold". The spaces $\mathbb{R}^n$, S^n, any open subsets of R^n are n-manifolds; the cartesian product of an n-manifold with a k-manifold is a (n+k)-manifold; so there are lots of manifolds. If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function [see 22M:133], then for almost all $a \in \mathbb{R}$, the solution set $W = \{w \in \mathbb{R}^n \ : \ f(w) = a\}$ is a (n-1)-manifold. So solution sets of equations often are manifolds. What about the other way around? Is every manifold actually realizable as a solution set in some $\mathbb{R}^n$? Maybe an easier question... Is every manifold homeomorphic to a subspace of some $\mathbb{R}^n$? Using Urysohn's lemma to develop the tool called *partitions of unity*, we obtain the following theorem.

Each compact n-manifold is homeomorphic to a subspace of some $\mathbb{R}^N$.

Of course the "N" depends on X; it is impossible to embed a high-dimensional manifold in a low-dimensional $\mathbb{R}^N$; in fact, generally $N \gneq n$. The embedding theorem in our text does not attempt to control the dimension N; however, the machinery of smooth manifolds (22M:133 or 22M:200) allows one to embed each compact n-dimensional smooth manifold in $\mathbb{R}^{2n}$.

Proof of Urysohn's Lemma. The text gives a detailed proof. So this discussion offers a different approach to some parts, and includes more comments and motivation for what we do. But the actual proof is close to the text's.

Somehow, we have to associate to each point $x \in X$ a number, $f(x)$. Furthermore, our function f has to be continuous [otherwise the proof would be trivial and the theorem would have no meaningful content], send set A to 0, and B to 1. All we know about X is the hypothesis that X is normal. Here is the plan: We are going to define a certain [large] collection of open sets in X; then we will decide for each $x \in X$ what $f(x)$ should be by looking at which of these open sets do, or do not, contain x.

Let D be the set of dyadic rationals in $[0,1]$, that is $D = \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{4}{8}, \frac{7}{8}, \dots\}$. We will construct a sequence of open sets U_q in X, indexed by $q \in D$.

First, let $U_1 = X$. Now we get to the real work:
Since X is normal, there exist disjoint open neighborhoods $U(A)$ and $V(B)$. Note that the existence of V disjoint from U tells us that $\bar{U} \cap B = \emptyset$, i.e. $\bar{U} \subseteq (X - B)$. Let U_0 be this neighborhood $U(A)$. For all the subsequent sets U_q that we will define, we will have $A \subseteq U_q$; and for all $q < 1$, $U_q \cap B = \emptyset$.

The closed set $\bar{U}_0$ is contained in the open set $X - B$. Since X is normal, there exists an open set (call it $U_{1/2}$) such that

$$\bar{U}_0 \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq (X - B) \ .$$

We continue inductively: interpolate $U_{1/4}$ between U_0 and $U_{1/2}$; interpolate $U_{3/4}$ between $U_{1/2}$ and $X - V$; then define $U_{1/8}$, $U_{3/8}$, etc.

We get a sequence of open sets U_q such that

(1) For each $q \in D$, $A \subseteq U_q$.
(2) $B \subseteq U_1$ and for each $q < 1$, $B \cap U_q = \emptyset$.
(3) For each $p, q \in D$ with $p < q$, we have $\bar{U}_p \subseteq U_q$.

We now define $f : X \to [0,1]$ by

$$f(x) = inf\{q \mid x \in U_q\}$$

for each $x \in X$.

The function f is defined because each point of X is contained in some U_q, at least in $U_1 = X$. By condition (1), f is 0 on set A. And by condition (2), f is identically 1 on set B. (Note: We are not claiming that f is 0 only on A, or 1 only on B. In general, the zero set and "1-set" of f will be larger than just A and B.) It remains to show that f is continuous. This is *Step 4* in the text's proof.

We first establish two mini-lemmas:

A. If $f(x) > q$ then $x \notin \bar{U}_q$.
B. If $f(x) < q$ then $x \in U_q$.

For each $x \in X$, let $D(x) = \{q \mid x \in U_q\}$. So $f(x) = inf\ D(x)$. The numbers q and the sets U_q are ordered the same way. So if $q \in D(x)$ and $q' > q$, then $q' \in D(x)$. The infimum $f(x)$ might be the smallest element of $D(x)$ or it might be a lower limit point that it not itself in $D(x)$.

Proof of A. If $f(x) > q$ then there must be some gap between q and $D(x)$; in particular, there exists some q' such that $q < q' < f(x)$. But $q' < f(x) \implies x \notin U_{q'}$, and then $\bar{U}_q \subseteq U_{q'} \implies x \notin \bar{U}_q$.

Proof of B. If $f(x) < q$ then there exists $q' \in D(x)$ such that $f(x) < q' < q$, in which case $q \in D(x)$, so $x \in U_q$.

We now can show that f is continuous. We need to show that the pre-image of each sub-basic set $(a, 1]$ or $[0, a)$ is open in X. Suppose first that $f(x) \in (a, 1]$. Pick some q with $a < q < f(x)$. We claim that the open set $W = X - \bar{U}_q$ is a neighborhood of x that is mapped by f into $(a, 1]$. First, by (A), $f(x) > q \implies x \in W$, so W *is* a neighborhood of x. If y is any point of W, then $f(y)$ must be $\geq q > a$; otherwise, if $f(y) < q$, then, by (B), $y \in U_q \subseteq \bar{U}_q$.

The argument is simpler for $f^{-1}[0, b)$. Suppose $f(x) < b$ and pick q such that $f(x) < q < b$. By (B), $x \in U_q$. We claim that the neighborhood U_q is mapped by f into $[0, b)$. Suppose y is any point of U_q. Then $q \in D(y)$, so $f(y) \leq q < b$.

HOMEWORK in Section 33
Due Wednesday Dec 12
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#1 (i.e. do you understand the proof of Urysohn's Lemma)
#2 (cute - see sample Final Exam problems below for another way to ask the same questions)
#3 (practice with metrics)
#7 (a review of LCH spaces. To warm up for this, you might first prove that a compact Hausdorff space is completely regular. Also remember our old friend the "pasting lemma".)

Problem. *(Sample problem for final exam)*

Write a one-page $\{$summary, outline, sketch$\}$ of a proof of Urysohn's Lemma.

Problem. *(Sample problem for final exam)*

a. *Suppose X is a countable space. Prove X has the Lindelöf property.*
b. *There exists [free gift, not done in our class] a space X that is countable, Hausdorff, and connected. Use this space to show that the theorem "regular + Lindelöf $\implies$ normal" cannot be modified to say that "Hausdorff + Lindelöf $\implies$ regular.*

(end of handout)