22M:132
Fall 07
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# $\mathbb{R}$ is Uncountable

The text gives a complete proof of this theorem. But the process is something that will come up again, so I want to make sure you understand what we are doing and why we are doing it. Here is a more "chatty" version of the proof.

**Remark.** We do not need all the properties of $\mathbb{R}$ to get a topological proof that the set is uncountable. The order-topology is not the interesting feature. Also the fact that $\mathbb{R}$ is connected is not relevant. What matters most is that each point of $X$ is a limit point; so around each point you can think of a "cloud" of other points hovering close. But around each of those, there is a "cloud", and pretty soon you start to think that the set might have to be uncountable.

**Theorem.** *Suppose $X$ is a compact Hausdorff space with the extra property that each point of $X$ is a limit point of $X$. That is,*

$$\forall x \in X \ , \ \forall U \text{ open such that } x \in U \ , \ \exists y \in U \text{ with } y \neq x \ .$$

*Then $X$ is uncountable.*

*Proof.* Let $A = \{a_1, a_2, \ldots\}$ be any countable subset of $X$. We will prove there exists a point of $X$ not included in the set $A$. To do this, we will construct a nested sequence $C_1 \supseteq C_2 \supseteq \ldots$ of closed sets with the property that $a_1 \notin C_1$, $a_2 \notin C_2$, …, $a_n \notin C_n$, etc. Any point $x$ in the intersection of all the $C_n$'s cannot be any one of the points $a_i$ of $A$. We will construct the nested sets so that each $C_n$ is nonempty and closed. Since the sets are nested, we know that any finite intersection of $C_n$'s is nonempty (since the intersection just equals the $C_n$ with the largest subscript). Since $X$ is compact, we can conclude that

$$\bigcap_{n=1}^{\infty} C_n \neq \emptyset \ .$$

which completes the proof.

*Step 1: Define set $C_1$.* Since each point of $X$ is a limit point, the neighborhood $X$ of $a_1$ must contain some other point of $X$; call that point $x_1$. Since $X$ is Hausdorff, there exist disjoint neighborhoods $U_1(a_1)$ and $V_1(x_1)$. Note that since $a_1$ has a neighborhood $U_1$ that misses $V_1$, we know that $a_1$ is not in the closure of $V_1$. We let $C_1$ be that closure, that is $C_1 = \bar{V}_1$.

*Step 2: Define set $C_2$.* We do not know how the point $a_2$ relates to the set $V_1$; for all we know, it might be that $a_2 \in V_1$ or maybe even $a_2 = x_1$. We want to work with a point $x_2 \in V_1$ that is different from $a_2$. Here is how to pick such a point:
If $a_2 \neq x_1$ then let $x_2 = x_1$. If $a_2 = x_1$ then use the fact that $x_1$ is a limit point of $X$ to say that the neighborhood $V_1(x_1)$ must contain some point of $X$ other than $x_1$

(i.e. other than $a_2$), and let $x_2$ be such a point. So now we have a point $x_2 \in V_1$ such that $x_2 \neq a_2$.

Since $X$ is Hausdorff, there exist disjoint open sets $U_2(a_2)$ and $V_2'(x_2)$. Let $V_2 = V_2'(x_2) \cap V_1$. Note that $V_2$ is nonempty [because of $x_2$], contained in $V_1$, and its closure does not contain $a_2$ [because of the neighborhood $U_2$]. Let $C_2 =$ the closure of $V_2$, that is $C_2 = \bar{V_2}$. Since $V_2 \subseteq V_1$, we have $\bar{V_2} \subseteq \bar{V_1}$, i.e. $C_2 \subseteq C_1$. Finally, note that $C_2$ misses $\{a_1, a_2\}$.

*Step n: Define set $C_n$.* The points $a_1, \ldots, a_n$ miss $V_n$; but we do not know how the point $a_{n+1}$ relates to the set $V_n$. Use the same argument as in the case $n = 2$ above to find an nonempty open set $V_{n+1}'$ whose closure misses $a_{n+1}$ and intersect that set with $V_n$ to get $V_{n+1}$ with the desired properties. Let $C_{n+1} = \bar{V}_{n+1}$.

Once we have the sequence $(C_n)$ as above, the [nonempty because $X$ is compact] intersection provides point[s] of $X - A$.

□

(End of handout)