22M:132
Fall 07
J. Simon

Comments on

The Converse of a Familiar Theorem on Compactness

One of our standard, familiar, important, basic theorems is:

Theorem. *If $f : X \to Y$ is a continuous surjection and X is compact, then Y is compact.*

How could there possibly be a converse?? The converse would be the audacious, and obviously false statement:

Theorem. *If $f : X \to Y$ is a continuous surjection and Y is compact, then X is compact.*

Can we add hypotheses to X, Y, or f to make such a statement true? First of all, we could map any [non-compact] space X to a single point $Y = \{y\}$. So let's put in the assumption that if we look at any individual point of Y, the pre-image is compact. That is, assume

- *For each point $y \in Y$, the pre-image $f^{-1}(y)$ is compact.*

This will not be enough.

Example. *Let $X = (0, 5\pi) \subseteq \mathbb{R}$. Let $f : X \to [-1, 1]$ by $f(x) = sin(x)$. Note that f is surjective and continuous; also for each $y \in [-1, 1]$, $f^{-1}(y)$ is a finite set, so $f^{-1}(y)$ is compact. But the range $[-1, 1]$ is compact, and the domain X isn't.*

Here is an additional hypotheses that will protect us from the above counterexample:

- *The function f is a closed map.*

In the above counter-example [to our dream theorem], the set $(0, Pi/2]$ is closed in X, but its image $(0, 1]$ is not closed in Y.

The additional hypothesis of *closed map* is enough to prove the theorem.

Theorem 1. *If $f : X \to Y$ is a continuous surjection that is a closed map, and Y is compact, then X is compact.*

To prove this theorem, we first recall (and extend) what we know about quotient maps.

Review quotient maps and quotient topology

If $f : X \to Y$ is any function, then there is an associated equivalence relation on X:

$$x \sim x' \iff f(x) = f(x') \ .$$

Note that he equivalence class $[x]$ is precisely $f^{-1}f(x)$. We call a set $A \subseteq X$ *saturated* if A is a union of equivalence classes. You can check the following:

Proposition. *For each $A \subseteq X$, A is saturated $\iff$ $A = f^{-1}(f(A)$.*

When X comes equipped with a topology, we define the *quotient topology* on Y by saying V is open in $Y \iff f^{-1}(V)$ is open in X. We say f is a *quotient map* if $f : X \to Y$ is surjective and Y has the quotient topology defined by f. Since V open $\implies f^{-1}(V)$ open, this topology on Y makes f continuous. Furthermore, the property "$f^{-1}(V)$ open in $X \implies V$ open in Y" tells us that if an open set $U \subseteq X$ is saturated, then $f(U)$ is open in Y. In summary, if $f : X \to Y$ is a quotient map, then

- V open in $Y \implies f^{-1}(V)$ is open in X;
- U is open in X and U saturated $\implies f(U)$ is open in Y.

Lemma. *If $f : X \to Y$ is a continuous closed map, then f is a quotient map*

Proof. We did this during our earlier class discussions of the quotient topology. □

Extending what we know about quotient maps

For a given set $A \subseteq X$, it is useful to analyze the set of all equivalence classes $[x]$ that are contained in A and the set of all equivalence classes $[x]$ that touch A.

Definition. *For a set $A \subseteq X$, the* saturation *of A is*

$$\mathit{sat}\, A = \bigcup \{[x] \ : \ [x] \cap A \neq \emptyset\}.$$

The saturated inside *of A is*

$$\mathit{inside}\, A = \bigcup \{[x] \ : \ [x] \subseteq A\}$$

Lemma 1.1. *Suppose $f : X \to Y$ is a continuous closed map.*

- *If C is closed in X then $sat\, C$ is closed in X.*
- *If U is open in X then $inside\, U$ is open in X.*

Proof. If C closed in X and f is a closed map, then $f(C)$ is closed in Y. Since f is continuous, $f^{-1}(C)$ is closed in X. But $f^{-1}(C) = sat\, C$.

Now suppose U is open in X We want to show $inside\, U$ is open. Check that $X - inside\, U = sat(X - U)$, which is closed by the first part of this theorem.

□

Proof of Theorem 1

We have $f : X \to Y$ surjective, continuous, and a closed map. Y is compact, and, for each $y \in Y$, $f^{-1}(y)$ is compact. Let $\{U_\alpha\}$ be an open cover of X. We seek a finite subcover of X.

Since each $f^{-1}(y)$ is compact, each set $f^{-1}(y)$ is contained in finitely many of the U_α. For each $y \in Y$, let $\mathcal{U}_y$ be some finite union of U_α's that covers $f^{-1}(y)$. Since each $\mathcal{U}_y$ is made of finitely many U_α's, if we knew that finitely many $\mathcal{U}_y$'s cover X, then we would have a finite number of finite numbers of U_α's covering X. So it suffices to show that X is covered by some finite number of the $\mathcal{U}_y$'s.

For each $y \in Y$, let $W_y = inside\, \mathcal{U}_y$. By Lemma 1.1, W_y is open. Since W_y is a saturated open set, and f is a quotient map, $f(W_y)$ is open in Y. Also W_y contains at least the equivalence class $f^{-1}y$, and f is surjective, so the collection $\{f(W_y)\ : y \in Y\}$ is an open cover of Y. Since Y is compact, there is a finite subcover $W_{y1}, \dots, W_{yn}$. Then the pullbacks $\{f^{-1}f(W_{y1}), \dots, f^{-1}f(W_{yn})\}$ cover X. But since the sets W_{yj} are saturated, each $f^{-1}f(W_{yj})$ is just W_{yj}. So this finite collection $W_{y1}, \dots, W_{yn}$ covers X. Thus the collection of larger sets $\{\mathcal{U}_{y1}, \dots, \mathcal{U}_{yn}\}$ covers X. □

[end of handout]