22M:132
Fall 07
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Sample Problems
for
Final Exam

NOTE. This list includes all problems, from all parts of the course, that will be used as the basis for the Final Exam. In particular, I have selected those problems from earlier material that seem most important to study for the Final. Also, I have deleted one of the sample problems from the most recent handout.

As always, the only promise is that "most" of the Exam will consist of problems taken from this list, and a problem might appear on the exam in slightly different from from the list. As announced in class, I expect approximately (2/3) of the Final Exam to be based on material since the last midterm (Problems 1-18 below) and (1/3) based on previous material (Problems 19-37, which are taken from the Sample Problems handout for Exam II).

Problem 1.

Suppose a compact Hausdorff space X has an open covering consisting of two sets $\{W_1, W_2\}$. Prove there exists an open covering $\{U_1, U_2\}$ of X such that for each j, $\bar{U}_j \subseteq W_j$.

Remark. This problem is not confined to manifolds; it is an application of normality that is used in the text's step 1 of proving there exists a partition of unity subordinate to the given covering of X.

Problem 2.

Write a 1-2 page summary of a proof that each compact m-manifold embeds in $\mathbb{R}^N$ for some N.

Problem 3.

Prove the statement or give a counterexample:
Suppose a space X can be written as a union $X = A \cup B$, where A and B are closed sets. Suppose for some space Y, we have continuous functions $f : X \to Y$ and $g : X \to Y$ such that $f|A$ is 1-1, and $g|B$ is 1-1. Then the function $f \times g : X \to Y \times Y$ given by $(f \times g)(x) = (f(x), g(x))$ is 1-1.

Problem 4.

Page 227 # 1
Prove that every manifold is regular and hence metrizable. (Hint: Recall "manifold" assumes Hausdorff and 2nd-countable; show that ever manifold is locally compact, hence regular.)

Problem 5.

Prove the following proposition, which says two versions of the definition of "manifold" are equivalent: *If each point of X has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^m$ then each point of X has an open neighborhood homeomorphic to $\mathbb{R}^m$.*

Problem 6.

Page 227 #3
Suppose X is Hausdorff and has the property that each point has a neighborhood homeomorphic with an open subset of $\mathbb{R}^m$. Assume further that X is compact. Sow X is an m-manifold. The point here is to use compactness + the "locally Euclidean" property to show that X has a countable basis.

Problem 7.

If $(X, \mathcal{T})$ is homeomorphic to a subspace of $(\mathbb{R}^\omega$, product topology), then $(X, \mathcal{T})$ is regular and has a countable basis.

Key steps:
  a. *There is a metric on $\mathbb{R}^\omega$ that gives the product topology.*
  b. *$\mathbb{R}^\omega$ in the product topology has a countable dense subset.*
  c. *A metric space with a countable dense subset has a countable basis for the topology.*
  d. *Each subspace of a 2nd-countable space is 2nd-countable.*
  e. *Each subspace of a metric space is metrizable.*
  e. *Each metric space is regular.*
  f. *Put the pieces together.*

Problem 8.

Write a one-page {summary, outline, sketch} of a proof of Urysohn's Lemma.

Problem 9.

    a. *Suppose X is a countable space. Prove X has the Lindelöf property.*

    b. *There exists [free gift, not done in our class] a space $X*$ that is countable, Hausdorff, and connected. Use this space to show that the theorem "regular + Lindelöf $\implies$ normal" cannot be modified to say that "Hausdorff + Lindelöf $\implies$ regular".*

Problem 10.   *Page 194 #5a*

Show that every metrizable space with a countable dense subset has a countable basis.

Problem 11.   *Page 194 #5b*

Show that every metrizable Lindelöf space has a countable basis.

Problem 12.   *Page 194 #6*

Show that $\mathbb{R}_\ell$ is not metrizable.

Problem 13.   *Page 194 #7*

Which of our four countability axioms (i.e.1st-countable, 2nd-countable, separable, Lindelöf) does S_Ω satisfy? What about $\bar{S}_\Omega$?

Problem 14.   *Page 194 #9*

Prove: The Lindelöf property is inherited by closed *subspaces*

Problem 15.   *Page 194 #10*

Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

(As an obvious hint, you can try first to prove this for the space $\mathbb{R}^\omega$ in the product topology.)

Problem 16.   *Page 194 #14*

Show that if X Lindelöf and Y is compact, then $X \times Y$ is Lindelöf .

Problem 17.

Prove: If X is a compact Hausdorff space with a countable basis, then there exists an embedding of X into $\mathbb{R}^\omega$, where $\mathbb{R}^\omega$ has the product topology.

Remark. This is a special case of the more general Urysohn metrization theorem. The proof is made easier by considering just spaces X that are compact.

Problem 18.

Write a one-to-two page proof:

If X is a regular space with a countable basis, then there exists an embedding of X into $\mathbb{R}^\omega$, where $\mathbb{R}^\omega$ has the uniform topology.

Remark. This is the general Urysohn metrization theorem.

Problem 19.

Let (f_n) be a sequence of functions, $f_n : X \to Y$, where X is a topological space and Y is a metric space; and let $f : X \to Y$ be some function.
  a. *Define the statement "(f_n) converges uniformly to f".*
  b. *Prove: If (f_n) converges uniformly to f and each f_n is continuous, then f is continuous.*
  c. *Prove: If X is compact, and (f_n) converges uniformly to f and each f_n is continuous and surjective, then f is surjective.*

Problem 20.

Suppose $f : X \to Y$ is a surjective continuous function, X is compact, and Y is Hausdorff. Prove that f is a quotient map.

Problem 21.

Suppose $f : X \to Y$ is continuous, 1-1, and surjective, X is compact, and Y is Hausdorff. Prove f is a homeomorphism.

Problem 22.

Show that the following properties of a space X are equivalent to each other:
  a. $X = A \cup B$, *where A and B are disjoint open sets.*
  b. $X = A \cup B$, *where A and B are disjoint closed sets.*
  c. $X = A \cup B$, *where neither of A, B intersects the closure of the other.*

Problem 23.

Suppose X is connected and $f : X \to Y$ is a surjective continuous map. Prove Y is connected.

Problem 24.

  a. *Define* path connected
  b. *Prove the continuous image of a path connected space is path connected.*
  c. *Prove: If X and Y are path connected, then $X \times Y$ is path connected. (State carefully whatever lemma[s] you use.)*

Problem 25.

Prove: If A is a connected subset of a space X, then the closure $\bar{A}$ is connected.

Problem 26.

(For this problem, assume we have proven that $\mathbb{R}$ is connected, and that finite products of connected spaces are connected.) The problem has two related parts.

Let $A = \{(a_n) \in \mathbb{R}^\omega \ : \ a_n = 0 \text{ for all but finitely many } n\}$. Here $\mathbb{R}^\omega$ has the product topology.

    i) *Show A is connected. (Hint: Write A as a nested union of connected sets.)*
    ii) *Use your result in Problem 25 to show that $\mathbb{R}^\omega$ is connected.*

Problem 27.

    a. *Use the fact that $[0,1]$ is connected, and appropriate theorem(s) about unions of connected sets to show that $\mathbb{R}^1$ [with the standard topology] is connected.*
    b. *Show that $\mathbb{R}^1_\ell$ is totally disconnected.*

Problem 28.

Suppose $f : S^1 \to \mathbb{R}$ is a continuous function.
Prove there exists $x \in S^1$ such that $f(x) = f(-x)$.

Problem 29.

    a. *Define* [connected] component
    b. *Prove that the components of any space are closed.*
    c. *Given an example to show that components do not have to be open.*

Problem 30.

    a. *Define* locally connected
    b. *Prove that X is locally connected if an only if for each open set $U \subseteq X$, each component of U is open,*

Problem 31.

    a. *Prove: If A is a closed subset of a compact space X then A is compact.*
    b. *Prove: If A is a compact subset of a Hausdorff space X, then A is closed in X.*

Problem 32.

    a. *If X is Hausdorff, $x \in X$, and $A \subseteq X$ is a compact set that does not contain x, then there exist disjoint neighborhoods U of x and V of A.*

    b. *If A, B are disjoint compact sets in a Hausdorff space X, then there exist disjoint neighborhoods U of A and V of B.*

Problem 33.

    a. *Give an example of an infinite collection of closed sets in $\mathbb{R}^1$ that has empty intersection but such that each finite subcollection has nonempty intersection.*

    b. *Prove that in a compact space X, any collection of closed sets with the finite intersection property has nonempty intersection.*

Problem 34.

Prove that a set $C \subseteq \mathbb{R}^n$ is compact if an only if C is closed and bounded.

(You may assume we know that $[0, 1]$ is compact.)

Problem 35.

    a. *Prove: If X is compact, then X is limit-point compact.*

    b. *Give an example of a space that is limit-point compact but not compact.*

Problem 36.

If X is a locally compact Hausdorff space and U is a neighborhood of a point x, then there exists a neighborhood V of x such that $\bar{V} \subseteq U$.

Problem 37.

Suppose X is a compact Hausdorff space and $\{A_1, A_2, \ldots\}$ is a countable collection of closed sets each having empty interior. Prove that the union $\bigcup_{n=1}^{\infty} A_n$ cannot be all of X.

Note: The same proof shows that a locally *compact Hausdorff space cannot be the union of countably many closed sets with empty interiors. Each complete metric space(every Cauchy sequence has a limit) has the same property. These are all versions of the* Baire Category Theorem*, Sec. 48. In fact, the complement of the closed sets is dense in X.*

This is enough. Happy studying!