

## Sample Problems for Exam II

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### Problem 1.

- Show that  $\mathbb{R}^\omega$  with the product topology is metrizable.
  - Show that  $\mathbb{R}^\omega$  with the box topology is not metrizable.
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### Problem 2.

Let  $X$  be a metric space, with metric  $d$ , and let  $A$  be a nonempty subset of  $X$ . Define a function  $d_A : X \rightarrow \mathbb{R}$  by

$$d_A(x) = \inf\{d(x, a) : a \in A\}$$

- Prove the function  $d_A$  is continuous.
  - Suppose  $A$  and  $B$  are disjoint closed sets in the metric space  $X$  and assume in addition that  $A$  is compact. Prove there exists  $\Delta > 0$  such that for all  $a \in A, b \in B$ ,  $d(a, b) \geq \Delta$ .
  - Give an example to show we cannot omit the assumption “ $A$  is compact” in part (b); that is, give an example of a metric space  $X$  with disjoint closed sets  $A, B$  such that  $\nexists$  a distance  $\Delta > 0$  between the sets.
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### Problem 3.

Let  $(f_n)$  be a sequence of functions,  $f_n : X \rightarrow Y$ , where  $X$  is a topological space and  $Y$  is a metric space; and let  $f : X \rightarrow Y$  be some function.

- Define the statement “ $(f_n)$  converges uniformly to  $f$ ”.
  - Prove: If  $(f_n)$  converges uniformly to  $f$  and each  $f_n$  is continuous, then  $f$  is continuous.
  - Prove: If  $X$  is compact, and  $(f_n)$  converges uniformly to  $f$  and each  $f_n$  is continuous and surjective, then  $f$  is surjective.
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### Problem 4.

- Define quotient map
- Explain why the following function is not a quotient map (here  $S^1$  is the unit circle in  $\mathbb{R}^2$ ):

$$f : [0, 2\pi) \rightarrow S^1 \text{ given by } f(t) = (\cos t, \sin t)$$

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**Problem 5.**

Suppose  $f : X \rightarrow Y$  is a surjective continuous function,  $X$  is compact, and  $Y$  is Hausdorff. Prove that  $f$  is a quotient map.

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**Problem 6.**

Suppose  $f : X \rightarrow Y$  is continuous, 1-1, and surjective,  $X$  is compact, and  $Y$  is Hausdorff. Prove  $f$  is a homeomorphism.

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**Problem 7.**

Let  $X = S^1$ , and define an equivalence relation on  $X$  by saying each point is equivalent to its antipode, i.e.

$$(x, y) \sim (-x, -y).$$

So each equivalence class consists of exactly two points. Let  $Y$  be the quotient space  $X/\sim$ .

Prove  $Y$  is homeomorphic to  $S^1$ .

*Hint:* Define a function  $f : Y \rightarrow S^1$  that takes each equivalence class to a single point of  $S^1$ . Then use previous problem(s?) to show  $f$  is a homeomorphism.

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**Problem 8.**

Show that the following properties of a space  $X$  are equivalent to each other:

- $X = A \cup B$ , where  $A$  and  $B$  are disjoint open sets.
  - $X = A \cup B$ , where  $A$  and  $B$  are disjoint closed sets.
  - $X = A \cup B$ , where neither of  $A, B$  intersects the closure of the other.
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**Problem 9.**

Suppose  $X = A_1 \cup A_2 \cup \dots$ , where each set  $A_i$  is connected and  $\forall i A_i \cap A_{i+1} \neq \emptyset$ .

Prove  $X$  is connected.

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**Problem 10.**

Suppose  $X$  is connected and  $f : X \rightarrow Y$  is a surjective continuous map. Prove  $Y$  is connected.

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**Problem 11.**

- a. Define path connected
  - b. Prove the continuous image of a path connected space is path connected.
  - c. Prove: If  $X$  and  $Y$  are path connected, then  $X \times Y$  is path connected. (State carefully whatever lemma[s] you use.)
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**Problem 12.**

Prove: If  $A$  is a connected subset of a space  $X$ , then the closure  $\bar{A}$  is connected.

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**Problem 13.**

(For this problem, assume we have proven that  $\mathbb{R}$  is connected, and that finite products of connected spaces are connected.) The problem has two related parts.

Let  $A = \{(a_n) \in \mathbb{R}^\omega : a_n = 0 \text{ for all but finitely many } n\}$ . Here  $\mathbb{R}^\omega$  has the product topology.

- i) Show  $A$  is connected. (Hint: Write  $A$  as a nested union of connected sets.)
  - ii) Use your result in part (i) to show that  $\mathbb{R}^\omega$  is connected.
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**Problem 14.**

Prove the interval  $[0, 1]$  is connected.

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**Problem 15.**

- a. Use Problems 9 and 14 to show that  $\mathbb{R}^1$  [with the standard topology] is connected.
  - b. Show that  $\mathbb{R}_\ell^1$  is totally disconnected.
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**Problem 16.**

Suppose  $f : S^1 \rightarrow \mathbb{R}$  is a continuous function.

Prove there exists  $x \in S^1$  such that  $f(x) = f(-x)$ .

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**Problem 17.**

Recall that a space  $X$  is locally path-connected if for each  $x \in X$  and each neighborhood  $U$  of  $x$ , there exists a neighborhood  $V$  of  $x$  such that  $V \subseteq U$  and  $V$  is path-connected.

Prove: If  $X$  is connected and locally path-connected, then  $X$  is path-connected. (More generally, if  $U$  is a connected open subset of a locally path-connected space, then  $U$  is path-connected.)

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**Problem 18.**

- a. Define [connected] component
  - b. Prove that the components of any space are closed.
  - c. Given an example to show that components do not have to be open.
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**Problem 19.**

- a. Define locally connected
  - b. Prove that  $X$  is locally connected if and only if for each open set  $U \subseteq X$ , each component of  $U$  is open,
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**Problem 20.**

- a. Prove: If  $A$  is a closed subset of a compact space  $X$  then  $A$  is compact.
  - b. Prove: If  $A$  is a compact subset of a Hausdorff space  $X$ , then  $A$  is closed in  $X$ .
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**Problem 21.**

- a. If  $X$  is Hausdorff,  $x \in X$ , and  $A \subseteq X$  is a compact set that does not contain  $x$ , then there exist disjoint neighborhoods  $U$  of  $x$  and  $V$  of  $A$ .
  - b. If  $A, B$  are disjoint compact sets in a Hausdorff space  $X$ , then there exist disjoint neighborhoods  $U$  of  $A$  and  $V$  of  $B$ .
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**Problem 22.**

Prove: If  $X$  and  $Y$  are compact spaces, then the product  $X \times Y$  is compact.

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**Problem 23.**

- a. Give an example of an infinite collection of closed sets in  $\mathbb{R}^1$  that has empty intersection but such that each finite subcollection has nonempty intersection.
  - b. Prove that in a compact space  $X$ , any collection of closed sets with the finite intersection property has nonempty intersection.
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**Problem 24.**

Prove the interval  $[0, 1] \subset \mathbb{R}^1$  is compact.

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**Problem 25.**

Prove that a set  $C \subseteq \mathbb{R}^n$  is compact if and only if  $C$  is closed and bounded.

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**Problem 26.**

Suppose a compact metric space  $X$  is expressed as the union of two open sets,  $X = U \cup V$ .

Prove there exists a number  $\lambda > 0$  such that each subset of  $X$  having diameter  $< \lambda$  is contained in  $U$  or in  $V$ .

*Remark: At the risk of pointing out the obvious, this is not claiming that all the small sets are in  $U$  or all the small sets are in  $V$ . Some small sets end up in  $U$ , others in  $V$ , and some in both. Saying that a set is contained in  $U$  does not prevent it from intersecting  $V$ , maybe even being contained in  $V$  as well as in  $U$ . If you think of a cartoon in which a small set starts inside  $U$  and moves gradually to escape from  $U$ , by the time a part of the small set gets outside  $U$ , the set is entirely contained in  $V$ .*

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**Problem 27.**

- a. Prove: If  $X$  is compact, then  $X$  is limit-point compact.
  - b. Give an example of a space that is limit-point compact but not compact.
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**Problem 28.**

- a. If  $X$  is Hausdorff,  $x \in X$ ,  $U$  a neighborhood of  $x$  such that the boundary  $\text{bd}U$  is compact, then there exists a neighborhood  $V$  of  $x$  such that the closure  $\bar{V} \subseteq U$ .
  - b. If  $X$  is a locally compact Hausdorff space and  $U$  is a neighborhood of a point  $x$ , then there exists a neighborhood  $V$  of  $x$  such that  $\bar{V} \subseteq U$ .
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**Problem 29.**

Suppose  $X$  is a compact Hausdorff space and  $\{A_1, A_2, \dots\}$  is a countable collection of closed sets each having empty interior. Prove that the union  $\bigcup_{n=1}^{\infty} A_n$  cannot be all of  $X$ .

*Note: The same proof works to show that a locally compact Hausdorff space cannot be the union of countably many closed sets with empty interiors. In fact, what one can show is that the complement of any countable collection of closed sets with empty interiors must itself be dense in  $X$ .*

**This is enough. Happy studying!**