22M:132
Fall 07
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# Manifolds

(Relates to text Sec. 36)

**Introduction.** Manifolds are one of the most important classes of topological spaces (the other is function spaces). Much of your work in subsequent topology courses (in particular 22M:133, 201, 200, 203) focuses on manifolds. (In Analysis and Differential Equations course you will study a lot of function spaces). The solution sets of equations are almost always manifolds. For example, for almost all values of $b$, the solution set of the equation $x^2 + y^2 = b^2$ is a circle in $\mathbb{R}^2$. More generally, for almost all values of the parameters $a, b, c, d, e, F$ (where "almost all" needs to be defined carefully) the solution sets for the equation $ax^2 + bxy + cy^2 + dx + ey = F$ are 1-dimensional manifolds (ellipses or hyperbolas; the parabolas are relatively rare) in $\mathbb{R}^2$. These sets have the property that when we look at them "up close", they look just line the real line. More formally, each point has an open neighborhood homeomorphic to an open interval in $\mathbb{R}$. Similarly, the solution sets for $ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + jz = K$ are (for almost all values of the parameters $a, b, \ldots, K$) surfaces in $\mathbb{R}^3$. These sets have the property that each point has an open neighborhood homeomorphic to an open 2-disk. In Figure 1, we see a topologically more complicated surface[1], a "double-torus", with three of these *coordinate patches* shown.

FIGURE 1. A surface showing some "coordinate patches"

[1]picture before drawing patches from *http://upload.wikimedia.org/wikipedia/commons/thumb/b/bc/Double-torus-illustration.png/569px-Double-torus-illustration.png*

**Definition.** *A* topological m-dimensional manifold *is a space $X$ such that each point $x \in X$ has an open neighborhood $U(x)$ that is homeomorphic to $\mathbb{R}^m$. (Equivalently, say $U(x)$ is homeomorphic to an open m-ball.)*

Usually, we insist also that $X$ is Hausdorff, and that it has some global countability or compactness property. Our text specifies that $X$ is Hausdorff and has a countable basis. Sometimes people use a seemingly weaker assumption, that $X$ is *paracompact* (see Section 41; paracompact is weaker than 2nd-countable in the sense that if $X$ is regular then 2nd-countable $\implies$ metric $\implies$ paracompact. On the other hand, regular + paracompact $\not\Rightarrow$ 2nd-countable, e.g $\mathbb{R}_\ell$). We will follow our text and only use the word *manifold* for spaces that are Hausdorff and second-countable; in fact, the main theorem we prove is for compact manifolds. In 22M:200 you may see the analogous embedding theorem for noncompact manifolds. [2]

**Why would we ask the question about existence of embeddings?** As noted above, manifolds arise naturally as subsets of $\mathbb{R}^N$ But manfolds arise in situations where it is not so obvious that they are subsets of some $\mathbb{R}^N$. For example, the quotient space of $S^2$ under the antipodal identification (recall this example from our study of quotient spaces) is a 2-manifold; but that does not make it clear that there is a subset of some $\mathbb{R}^N$ homeomorphic to this manifold. What about $\mathcal{G}_{4,2}$ = set of all 2-dimensional subspaces of $\mathbb{R}^4$. What about the quotient space of a regular octagon disk with edges identified as shown below:

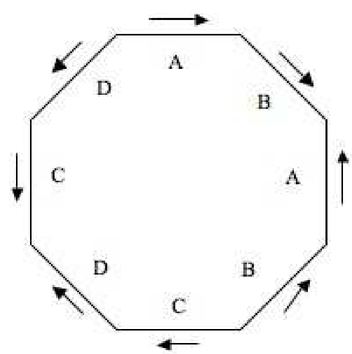


FIGURE 2. Identify edges to each other according to the labels (and directions) to get a manifold.

**Theorem** (Text theorem 36.2)**.** *If $X$ is a compact m-manifold, then $X$ is homeomorphic to a subspace of some $\mathbb{R}^N$. In other words, $X$* embeds *in some Euclidean space $\mathbb{R}^N$.*

*Remark.* In the following proof, the dimension $N$ depends on the dimension $m$ AND the number of chart neighborhoods needed to cover the manifold. In fact, all that matters is the dimension $m$ (see Cor. 50.8 later in the text).

[2]See also the comments on second-countable vs. paracompact for manifolds in *http : //en.wikipedia.org/wiki/Topological[underscore]manifold.*

*Remark.* In the proof, we will use a bunch of Urysohn functions: we will have certain disjoint closed sets and need functions from $X \to \mathbb{R}$ to separate them. The text does this by noting that $X$ is normal and invoking Urysohn's lemma. One can also get the desired functions by using the fact that $X$ is a manifold to define the desired functions using systems of concentric $m$-balls.

*Proof of Theorem.*

**Step 1. Construct a collection of m-balls covering $X$ and Urysohn functions that emphasize these m-balls.** Since $X$ is an m-manifold, each point $x \in X$ has an open neighborhood $U(x)$ homeomorphic to $\mathbb{R}^m$. Let $f_x : U(x) \to \mathbb{R}^m$ be a homeomorphism. By composing $f_x$ with a translation of $\mathbb{R}^m$, we may assume $f_x(x) = \vec{0}$. Let $A_x = f_x^{-1}(B(\vec{0}, 1))$. The closure of $A_x$ in $X$, $\bar{A}_x$, is just the set $f_x^{-1}$(closed ball $\bar{B}(\vec{0}, 1)$), since that pre-image set is compact and hence closed in $X$. Similarly, let $B_x = f^{-1}(B(\vec{0}, 2))$. The set $B_x$ is open in $U(x)$, hence open in $X$, so $(X - B_x)$ is closed in $X$ and disjoint from $\bar{A}_x$. Let $\phi_x : X \to [0, 1]$ be a Urysohn function that is 1 on $\bar{A}_x$, 0 on $(X - B_x)$.

**Step 2. Pick a finite sub-cover** The sets $\{A_x\}$ are an open cover of the compact space $X$, so there exists a finite subcover, $\{A_{x1}, A_{x2}, \ldots, \mathcal{A}_{xn}\}$. For tidiness, just label these [subsets of $X$ homeomorphic to] balls $A_1, A_2, \ldots, \mathcal{A}_n$.

**Step 3. Note properties of the sets and maps.** Let $\{f_1, f_2, \ldots, f_n\}$ be the corresponding homeomorphisms from $U_{xj} \to \mathbb{R}^m$. Let $\phi_j$ be the Urysohn function from **Step 1** that is 1 on $A_j$ and 0 outside $B_j$. Note that the restriction of $f_j$ to $A_j$ is an embedding. If we multiply an embedding of a set into $\mathbb{R}^m$ by a positive scalar, we still get an embedding.

**Step 4. Use the embeddings and Urysohn maps to construct maps from $X \to \mathbb{R}^m$ that are embeddings on parts of $X$.** We want to construct, for each $j = 1, 2, \ldots, n$, a map from $X \to \mathbb{R}^m$ that is an embedding on each of the balls $A_j$. Consider the function $\phi_j * f_j : U_j \to \mathbb{R}^m$. Since $\phi_j$ is defined on all $X$ and $f_j$ is defined on $U_j$, this product makes sense and is continuous on $U_j$. Furthermore, it is an embedding on $A_j$, a little mysterious but still continuous through the rest of $B_j$, and then 0 on $U_j - B_j$. We can then extend $\phi_j * f_j$ continuously to all of $X$ by making the function identically 0 on $X - B_j$. Let $h_j$ denote this extension. For each $j$,we have $h_j : X \to \mathbb{R}^m$ is a continuous function that is an embedding on $A_j$ and 0 outside $B_j$.

**Step 5. Combine the maps $\{h_j\}$ to get one function from $X$ into $\mathbb{R}^N$.** *(This is the key trick.)*
Define a function $H : X \to (\mathbb{R}^m)^n = \mathbb{R}^{mn}$ by $H(y) = (h_1(y), \ldots, h_n(y))$. The function $H$ is a cartesian product of continuous functions, so $H$ is continuous. Since $X$ is compact and $\mathbb{R}^{mn}$ is Hausdorff, we would know $H$ is an embedding (and so be done) if we could show that $H$ is $1 - 1$. We can accomplish this with one extra step: Use the Urysohn functions $\phi_j$ to help distinguish points of $X$ from each other.

Define $F : X \to (\mathbb{R} \times \ldots \times \mathbb{R}) \times (\mathbb{R}^m \times \ldots \times \mathbb{R}^m) = \mathbb{R}^{n(m+1)}$ by

$$F(y) = (\phi_1(y), \ldots, \phi_n(y), h_1(y), \ldots, h_n(y)) \ .$$

**Step 6. Show $F$ is an embedding.** The function $F$ is a product of continuous maps, so $F$ is continuous. Since $X$ is compact (and $\mathbb{R}^N$ is Hausdorff) it suffices to show that $F$ is $1-1$. Let $x, y \in X$ with $x \neq y$. We will show that $F(x) \neq F(y)$ by finding an index $j$ for which **at least one of** $\phi_j$ or $h_j$ distinguishes $x$ from $y$.

The balls $\{A_j\}$ cover $X$, so there exists $j$ with $x \in A_j$. Thus $\phi_j(x) = 1$. If $\phi_j(y) \neq 1$, then $F(x) \neq F(y)$. [This is why we added the extra $n$ coordinates to $H$ in constructing $F$.] If $\phi_j(y) = 1$ then $y \in B_j$ since $\phi_j = 0$ on $X - B_j$. Now see what the function $h_j$ does to $x$ and $y$. Since $x$ and $y$ are contained in $U_j$, $h_j$ is given by the recipe $h_j(x) = \phi_j(x) * f_j(x)$ and $h_j(y) = \phi_j(y) * f_j(y)$; since $\phi_j(x) = 1 = \phi_j(y)$, this says $h_j(x) = f_j(x)$ and $h_j(y) = f_j(y)$. But the homeomorphism $f_j$ is injective on $U_j$, so $x \neq y \implies f_j(x) \neq f_j(y) \implies F(x) \neq F(y)$.

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## Sample problems for Final Exam

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**Problem 1.**

*Suppose a compact Hausdorff space $X$ has an open covering consisting of two sets $\{W_1, W_2\}$. Prove there exists an open covering $\{U_1, U_2\}$ of $X$ such that for each $j$, $\bar{U}_j \subseteq W_j$.*

Remark. This problem is not confined to manifolds; it is an application of normality that is used in the text's step 1 of proving there exists a partition of unity subordinate to the given covering of $X$.

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**Problem 2.**

*Write a 1-2 page summary of a proof that each compact $m$-manifold embeds in $\mathbb{R}^N$ for some $N$.*

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**Problem 3.**

*Suppose a space $X$ (assume $X$ is compact and Hausdorff, even metric if you want) can be expressed as the union of two open sets $U \cup V$, where $U$ is homeomorphic to an open subset of $\mathbb{R}^2$ and $V$ is homeomorphic to an open subset of $R^2$. Prove that there exists an embedding of $X$ into $\mathbb{R}^6$.*

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**Problem 4.**

Prove the statement or give a counterexample:
*Suppose a space $X$ can be written as a union $X = A \cup B$, where $A$ and $B$ are closed sets. Suppose for some space $Y$, we have continuous functions $f : X \to Y$ and $g : X \to Y$ such that $f|A$ is 1-1, and $g|B$ is 1-1. Then the function $f \times g : X \to Y \times Y$ given by $(f \times g)(x) = (f(x), g(x))$ is 1-1.*

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**Problem 5.**

*Page 227 # 1*
*Prove that every manifold* (recall "manifold" assumes Hausdorff and 2nd-countable) *is [locally compact, hence] regular and hence metrizable.*

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**Problem 6.**

Prove the following proposition, which says two versions of the definition of "manifold" are equivalent: *If each point of $X$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^m$ then each point of $X$ has an open neighborhood homeomorphic to $\mathbb{R}^m$.*

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**Problem 7.**

*Page 227 #3* The point here is to use compactness + the "locally Euclidean" property to show that $X$ has a countable basis.

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**Problem 8.**

Prove the statement or give a counterexample:
*Suppose a space $X$ can be written as a union $X = A \cup B$, where $A$ and $B$ are closed sets. Suppose for some space $Y$, we have continuous functions $f : X \to Y$ and $g : X \to Y$ such that $f|A$ is 1-1, and $g|B$ is 1-1. Then the function $f \times g : X \to Y \times Y$ given by $(f \times g)(x) = (f(x), g(x))$ is 1-1.*

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*(end of handout)*