22M:132
Fall 07
J. Simon

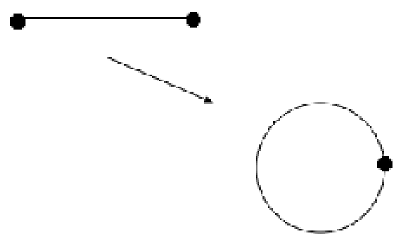
Comments on

Quotient Spaces and Quotient Maps

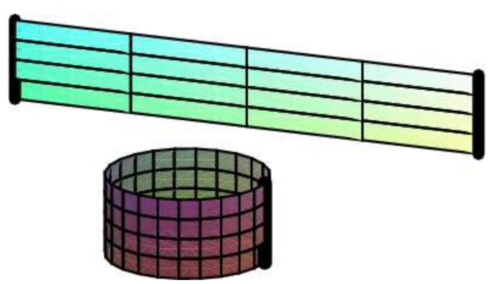
There are many situations in topology where we build a topological space by starting with some (often simpler) space[s] and doing some kind of " gluing" or "identifications". The situations may look different at first, but really they are instances of the same general construction. In the first section below, we give some examples, without any explanation of the theoretical/technial issues. In the next section, we give the general definition of a *quotient space* and examples of several kinds of constructions that are all special instances of this general one.

1. Examples of building topological spaces with interesting shapes by starting with simpler spaces and doing some kind of gluing or identifications.

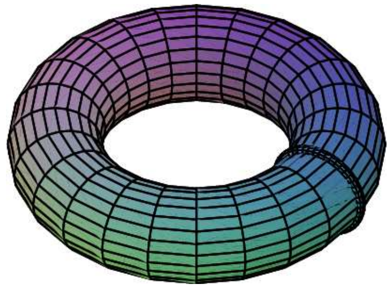
Example 0.1. *Identify the two endpoints of a line segment to form a circle.*



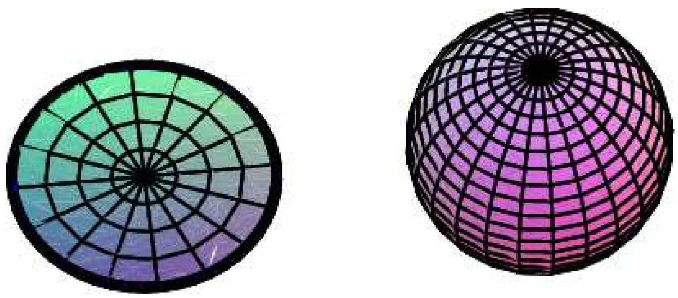
Example 0.2. *Identify two opposite edges of a rectangle (i.e. a rectangular strip of paper) to form a cylinder.*



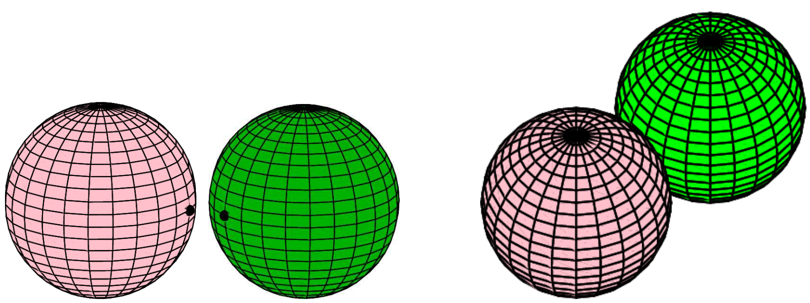
Example 0.3. *Now identify the top and bottom circles of the cylinder to each other, resulting in a 2-dimensional surface called a* torus*.*



Example 0.4. *Start with a round 2-dimensional disk; identify the whole boundary circle to a single point. The result is a surface, the 2-sphere.*



Example 0.5. ?? *Take two different spheres (in particular, disjoint from each other); pick one point on each sphere and glue the two spheres together by identifying the chosen point from each.*



Example 0.6. *The set of all homeomorphisms from a space X onto itself forms a group: the operation is composition* (The composition of two homeomorphisms is a homeomorphism; functional composition is associative; the identity map $I : X \to X$ is the identity element in the group of homeomorphisms.) *The group of all self-homeomorphisms of X may have interesting subgroups. When we specify some [sub]group of homeomorphisms of X that is isomorphic to some abstract group G, we call this an* **action** *of the group G on X. Note that in this situation, we are viewing the elements of G as homeomorphisms of X, and the group operation $\circ$ in G as function composition; so, in particular, for each $g, h \in G$, $x \in X$, we are insisting that $(g \circ h)(x) = g(h(x))$.*
When we have a group G acting on a space X, there is a "natural" quotient space. For each $x \in X$, let $Gx = \{g(x) \mid g \in G\}$. View each of these "orbit" sets as a single point in some new space X^*.

2. Definition of *quotient space*

Suppose X is a topological space, and suppose we have some equivalence relation "$\sim$" defined on X. Let X^* be the set of equivalence classes. We want to define a special topology on $X*$, called the *quotient topology.* To do this, it is convenient to introduce the function

$$\pi : X \to X^*$$

defined by

$$\pi(x) = [x] \ ,$$

that is, $\pi(x) =$ the equivalence class containing x.This can be confusing, so say it over to yourself a few times: π is a function from X into the power set of X; it assigns to each point $x \in X$ a certain subset of X, namely the equivalence class containing the point x. Since each $x \in X$ is contained in exactly one equivalence class, the function $x \to [x]$ is well-defined. At the risk of belaboring the obvious, since each equivalence class has at least one member, the function π is surjective.

Before talking about the quotient *topology*, let's look at several examples of the quotient sets X^*.

Example 0.7. *Let $X = \{a, b, c\}$, a set with 3 points. Partition X into two equivalence classes: $\{1, 3\}$ and $\{2\}$. So X^* has two elements, call them O and T. The function π is*

$$\pi(1) = O, \pi(2) = T, \pi(3) = O \ .$$

Example 0.8. *In our previous example 0.1, one equivalence class has two elements; every other equivalence class is a singleton. Likewise, in example* **??***, one equivalence class has two elements and all the others are singletons.*

Example 0.9. *In our previous example 0.2, each equivalence class coming from points on the vertical edges being identified consists of two points; all other equivalence classes are singletons.*

Example 0.10 (shrinking a set to a point)**.** *Let X be any space, and $A \subseteq X$. The the quotient space X/A is the set of equivalence classes $[x]$, where $[x] = A$ if $x \in A$ and $[x] = \{x\}$ if $x \notin A$. The set X^* has one "giant" point A and the rest are just the points of $X - A$. This is the situation in example 0.4*

Example 0.11. *Let X be the real line $\mathbb{R}^1$. Let G be the additive group of integers, Z. Define an action of G on $\mathbb{R}$ by $n(x) = x + n$ for each $n \in \mathbb{Z}, x \in \mathbb{R}$. Here the set Gx consists of all integer translates of the point x. Note that the sets Gx do form a partition of X. That is, the relation $x \sim y \iff$ there exists $g \in G$ with $g(x) = y$ is an equivalence relation on X. [Unassigned exercise: Check this claim; it depends on the fact that G is a group.]*

However, unlike our previous examples, it may be not so obvious what a geometric "picture" of X^* looks like: the number of points is the same as the half-open interval $[0, 1)$; but what should the topology be??

3. The quotient topology

If we think of constructing X^* by actually picking up a set X and squishing some parts together, we would like the passage from X to X^* to be continuous. We make this precise by insisting that the projection map

$$\pi : X \to X^* \qquad \pi(x) = [x]$$

be continuous. This puts an obligation on the topology we assign to X^*: If a set U is open in X^* then $\pi^{-1}(U)$ is open in X. (Think of this as an "upper bound" on which sets in X^* can be open.) We define the *quotient topology* on X^* by letting all sets U that pass this test be admitted.

A set U is open in X^* if and only if $\pi^{-1}(U)$ is open in X. The quotient topology on X^* is the finest topology on X^* for which the projection map π is continuous.

We now have an unambiguously defined special topology on the set X^* of equivalence classes. But that does not mean that it is easy to recognize which topology is the "right" one. Going back to our example 0.6, the set of equivalence classes (i.e. orbit sets Gx) is in 1-1 correspondence with the points of the half-open interval $[0, 1)$. But that does not imply that the quotient space, with the quotient topology, is homeomorphic to the usual $[0, 1)$. To understand how to recognize the quotient spaces, we introduce the idea of *quotient map* and then develop the text's Theorem 22.2. This theorem may look cryptic, but it is the tool we use to prove that when we think we know what a quotient space looks like, we are right (or to help discover that our intuitive answer is wrong).

4. Quotient maps

Suppose $p : X \to Y$ is a map such that

a . p is surjective,
b . p is continuous [i.e. U open in $Y \implies p^{-1}$ open in X], and
c . $U \subseteq Y$, $p^{-1}(U)$ open in $X \implies U$ open in Y.

In this case we say the map p is a *quotient map.*
The last two items say that U is open in Y if and only if $p^{-1}(U)$ is open in X.

Theorem. *If $p : X \to Y$ is surjective, continuous, and an open map, the p is a quotient map. If $p : X \to Y$ is surjective, continuous, and a closed map, then p is a quotient map.*

Proof. The proof of this theorem is left as an unassigned exercise; it is not hard, and you should know how to do it. (Consider this part of the list of sample problems for the next exam.) □

Remark. *Note that the properties "open map" and "closed map" are independent of each other (there are maps that are one but not the other) and strictly stronger than "quotient map" [HW Exercise 3 page 145].*

Remark (Saturated sets)**.** *A quotient map does not have to be an open map. But it does have the property that* certain *open sets in X are taken to open sets in Y. We say that a set $V \subset X$ is* saturated *with respect to a function f [or with respect to an equivalence relation $\sim$] if V is a union of point-inverses [resp. union of equivalence classes]. A quotient map has the property that the image of a saturated open set is open. Likewise, when defining the quotient topology, the function $\pi : X \to X^*$ takes saturated open sets to open sets.*

Suppose now that you have a space X and an equivalence relation $\sim$. You form the set of equivalence classes X^* and you give X^* the quotient topology. How can you know what the space X^* looks like? Let's use the group action example 0.6 to illustrate how we can answer the question.

The interval $[0, 1)$ has the "right number" of points. So does the circle S^1. Which is really the quotient space? It turns out S^1 is the right answer. To see this, define a map $q : \mathbb{R} \to S^1$ by $q(t) = \langle cos(2\pi t), sin(2\pi t)\rangle \in \mathbb{R}^2$. We claim this function q has two properties; together, they imply (Theorem 22.2) that the quotient space of $\mathbb{R}$ under the action of $\mathbb{Z}$ is homeomorphic to S^1.

a . The function q distinguishes points of the domain $\mathbb{R}$ exactly the same way as the equivalence relation $\sim$. That is, $q(x) = q(y) \iff [x] = [y]$.
b . The map q is a quotient map.

Proposition. *In the preceding example (action of $\mathbb{Z}$ on $\mathbb{R}$), X^* is homeomorphic to S^1.*

Proof. Define a function $f : S^1 \to X^*$ as follows: For each $z \in S^1$, $f(z) = \pi(q^{-1}(z))$. Because of condition (a) above, the function f is well-defined and is a bijection between S^1 and X^*.
We next show f is continuous. Let U be an open set in X^*. Then $\pi^{-1}(U)$ is an open set in $\mathbb{R}$ that is saturated with respect to π. By condition (a), since π and q have exactly the same point-inverses, $\pi^{-1}(U)$ is also saturated with respect to q. But then, since q is a quotient map, $q(\pi^{-1}(U))$ is open in S^1. Since $f^{-1}(U)$ is precisely $q(\pi^{-1}(U))$, we have that $f^{-1}(U)$ is open. The proof that f^{-1} is continuous is almost identical.

□

There is one case of quotient map that is particularly easy to recognize. Once we study *compact* spaces, we will have the following:

Theorem. *Suppose $f : X \to Y$ is a continuous surjective function. If X is compact and Y is Hausdorff, then f is a quotient map.*

Proof. We show that f is a closed map. Let C be a closed subset of X. A closed subset of a compact space is compact, so C is compact. The continuous image of a compact set is compact, so $f(C)$ is a compact subset of Y. A compact subset of a Hausdorff space is closed. So $f(C)$ is closed in Y. □

This theorem tells us that in all of our examples above (except 0.6, where we needed a fancier proof), we have the "right" picture of the quotient space.

[end of handout]