22M:132
Fall 07
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Information and Sample Problems for Exam I

Instructions. This is a "closed book" exam; you should have no books, scratch paper, or other written material available during the exam. Do all your work in the exam booklet provided.
***There are 8 problems; each is worth 10 points, so the total is 80 points.

Advice. The exam is written under the assumption that you know the material and can work the problems efficiently. So if you get stuck on something, don't spend a lot of time on it until after you've worked the problems you can do more quickly.

NOTE NOTE NOTE. The first 8 problems are an actual exam I gave previously in 22M:132. For our exam, I do not expect to include the topic of "well-ordering" (so Problem 2 below is moot). But students asked to see an "actual" exam - so here it is.

The rest of the problems... are samples. I may send out more sample problems later (e.g. Friday or Saturday). My standard commitment is that "most of the problems on the exam will be taken from the list of sample problems slight modifications of the samples". I reserve the right to have problems on the exam that are not represented in the samples.

Problem 1. *It took some work to prove that $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Q}$ are countable. Using these facts, and whatever general theorems you need about countable sets, explain how we know that each of the following sets A is countable.*

- *a)* $A = \{(p, q) \in \mathbb{Q} \times \mathbb{Q}$ *such that* $p < q\}$
- *b)* $A = \mathbb{Z} \times \{1, 2\}$.
- *c) Let* $r\mathbb{Z}$ *denote the set of all multiples* $r \cdot z$, $z \in \mathbb{Z}$. *Define*

$$A = \pi\mathbb{Z} \cup \pi^2\mathbb{Z} \cup \pi^3\mathbb{Z} \cup \pi^4\mathbb{Z} \cup \dots$$

Problem 2. *Suppose X is an uncountable set with a well-ordering "$<$". Show there exists a subset $Y \subseteq X$ with the following property:*

$$\forall y \in Y, \{s \in Y \mid s < y\} \text{ is countable.}$$

Problem 3. *Suppose* $(X, \mathcal{T})$ *and* $(Y, \mathcal{T}')$ *are topological spaces. Let* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$ *be the coordinate projection functions, that is*

$$\pi_X(x, y) = x \quad \text{and} \quad \pi_Y(x, y) = y.$$

Prove that the product topology is the coarsest topology on $X \times Y$ *that makes both projections continuious.*

Problem 4. *Let* $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$ *with the product topology. Let* $A = (0, 1) \times (0, 1) \subseteq X$.

(a) Find int(A).

(b) Find the set of limit points of A.

(c) Find $\overline{A}$.

(d) Find bd(A).

Problem 5. *Let* $(X, \mathcal{T})$ *be a topological space, with* $A \subseteq X$. *Prove:*

$$bd(A) = \emptyset \iff A \text{ is both open and closed.}$$

(Make it clear what definition or theorem(s) you are using.)

Problem 6. *Let* $(X, \mathcal{T})$ *be a topological space, with* $Y \subseteq X$*; give* Y *the subspace topology* $\mathcal{T}_Y$. *Here are two* false *"theorems". Give counterexamples to the "theorems" as stated, and change the statements to make them valid theorems.*

(a) If $U \subseteq Y$ *is open in* Y *(i.e.* $U \in \mathcal{T}_Y$*), then* U *is open in* X.

(b) If $C \subseteq Y$ *is closed in* Y *then* C *is closed in* X.

Problem 7. *Suppose* $(X, <)$ *and* $(X', <')$ *are simply ordered (i.e. linearly ordered) sets. Give each set the order topology. Suppose* $f : X \to X'$ *is surjective and monotonic, that is*

$$\forall x, y \in X,\ x < y \implies f(x) <' f(y).$$

Show that f *is a homeomorphism.*

Problem 8. *We are going to define an unusual topology on the set of natural numbers, $\mathbb{Z}_+$, by specifying a basis. For each $j \in \mathbb{Z}_+$, let*

$$U_j = \{n \in \mathbb{Z}_+ \mid n \text{ is divisible by } j\} .$$

For example, $U_3 = \{3, 6, 9, 12, \dots\}$.

- *(a) Show the set of these U_n's is a basis for a topology on $\mathbb{Z}_+$. (Hint: first observe that $r \in U_p \implies U_r \subseteq U_p$.)*
- *(b) Show the topology is not Hausdorff.*

END OF EXAM FROM A PRIOR YEAR

ADDITIONAL SAMPLE PROBLEMS

Problem 9. Definitions: *You should know all the definitions we have discussed in class (more precisely, all the definitions in all the sections of the text we have covered). The specific topologies you should be able to recognize/work with are:*

- *The usual (i.e. standard) topology on $\mathbb{R}^n$*
- *The discrete and indiscrete topologies*
- *The order topology on an ordered set*
- *The product topology (finite or infinite products)*
- *The box topology on an infinite product*
- *The lower-limit topology on $\mathbb{R}$. When we want to use this, we write $\mathbb{R}_\ell$.*
- *The counter-finite (i.e. finite-complements) topology on $\mathbb{R}$.*

Problem 10. *(list of "HW" problems from Section 19)*

- *Page 118 Ex 2*
- *Page 118 Ex 6*
- *Page 118 Ex 7*
- *Page 118 Ex 10a (read b), c*

Problem 11. *Assuming Theorems 7.1, 7.2, and 7.3, be able to prove various sets are countable. This includes specific sets such as any part(s) of Page 51 Ex 5, and also general results such as*

- *Theorem 7.6*
- *Theorem 7.7*

Problem 12. *(this proposition, and the next, are how I think we should navigate through Section 13)*

If $\mathcal{B}$ is a collection of subsets of X satisfying the two properties in the Definition on page 78, then the set of all unions of elements of $\mathcal{B}$ is a topology for X.

Problem 13. (and for subbases...)
If $\mathcal{S}$ is a collection of subsets of X whose union is all of X, then the set of all unions of finite intersections of elements of $\mathcal{S}$ is a topology for X.

Problem 14.
- *Page 83 Ex 1*
- *Page 83 Ex 4 a,b*
- *Page 83 Ex 7*
- *(variation on Page 83 Ex 6) Decide whether or not the lower-limit topology on $\mathbb{R}$ and the finite-complements topology on $\mathbb{R}$ are comparable. Explain your answer.*

Problem 15. *Suppose $(X, \mathcal{T})$is a topological space and $\mathcal{S}$ is a countable subbasis for $\mathcal{T}$. Decide whether or not there must exist a countable basis for $\mathcal{T}$; prove or give counter-example.*

Problem 16. *If $X \times Y$ has the product topology, prove*

- *(a) The projection $\pi_X : X \times Y \to X$ is continuous.*
- *(b) The projection $\pi_X : X \times Y \to X$ is an open map.*

Problem 17. *State and prove (or give counterexample[s]) theorems analogous to the previous ones, but now for infinite products with the poduct topology; for infinite products with the box topology.*

Problem 18.
- *Page 92 Ex 8*
- *Page 92 Ex 9*

Problem 19. *Using the text definition of* closure*, prove that a set $A \subseteq X$ is closed $\iff$ A contains all of its limit points.* (This is an immediate corollary of Theorem 17.6, so don't just quote that theorem; view this problem as an alternate way to state Theorem 17.6 and prove it "from scratch".)

Problem 20. *Page 95 Theorem 17.4*

Problem 21. *Page 101 Exercises 6, 8, 9*

Problem 22. *Prove that $X \times Y$ is Hausdorff $\iff$ each of X and Y is Hausdorff.*

Problem 23. *Decide if the theorem in the previous theorem still works for infinite products in the product topology? in the box topology?? (so there are 4 situations to decide: implication "if" or implication "only if", and using one of he two topologies for the product).*

Problem 24. *Page 102 Ex 19*

Problem 25. *Page 102 Ex 20 (where one might use different sets for A, B)*

[End of list as of today 9/24/07. Missing from the list so far is Section 18. There are LOTS of possible problems from this section; I hope to post those by the end of the week.]