22M:132
Fall 07
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Handout 2

Comments and Homework in Sections 16 and 17

First a comment on "basis". A collection $\mathcal{B}$ of subsets of X is *a basis for a topology on* X if the elements of $\mathcal{B}$ cover X and the intersection of any two elements of $\mathcal{B}$ is a union of elements of $\mathcal{B}$. This is a definition.

On the other hand, suppose we have a set X **with** a given topology $\mathcal{T}$. A collection $\mathcal{B}$ of subsets of X is *a basis for* $\mathcal{T}$ if $\mathcal{B} \subseteq \mathcal{T}$ [i.e. $\mathcal{B}$ consists of open sets] and each element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Here is a proposition to reassure us that these two uses of the word *basis* are essentially the same.

Proposition. *Suppose $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ as in the second definition above. Then $\mathcal{B}$ is a "basis for a topology" as in the first definition, and the topology generated by $\mathcal{B}$ is precisely $\mathcal{T}$.*

Proof. "Exercise for the reader" [not to hand in]. □

Section 16: The Subspace Topology

In this section, we will talk about various sets with various topologies, sometimes one set with two different topologies. To help this discussion, we will use the notation (see page 76) $(X, \mathcal{T})$ for a set X equipped with a given topology $\mathcal{T}$. The pair $(X, \mathcal{T})$ is called a *topological space.*

Lemma (16.0)**.** *Suppose $(X, \mathcal{T})$ is a topological space, and $Y \subseteq X$. Then*

$$\{U \cap Y \mid U \in \mathcal{T}\}$$

is a topology on Y.

Definition. *This topology on Y is called the* subspace topology *or* relative topology *or* induced topology. We sometimes denote the subspace topology on a subset Y induced by a topology $\mathcal{T}$ on X as $\mathcal{T}_Y$.

Proof. The text states and proves Lemma (16.0) as remarks after the definition.

□

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Key properties of the subspace topology are given in the text (and more in later sections). For now, the key facts are:

Proposition (16.1)**.** *If $\mathcal{B}$ is a basis for $\mathcal{T}$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.*

Proposition (16.2)**.** [READ this carefully, and convince yourself that this says the same thing as the text does for (16.2)]. *If $U \in \mathcal{T}_Y$ and $Y \in \mathcal{T}$ then $U \in \mathcal{T}$.*

Proposition (16.3)**.** *Subspace topology behaves well with respect to cartesian product.*

The subspace topology does not behave so simply in the case of an ordered set. If X is an ordered set and $Y \subseteq X$, then the ordering on X defines an ordering on Y. So we have two "natural" topologies on Y: The subspace topology $\mathcal{T}_Y$ and the order topology, which we denote $\mathcal{T}_{\text{ord}}$.

Proposition (16.4)**.** *If Y is* convex *in the ordering on X, then $\mathcal{T}_Y = \mathcal{T}_{\text{ord}}$.*

Optional problem. Without the assumption that Y is convex, is it still true that one of the topologies contains the other? Prove or give counterexample.

Homework Due Wednesday Sept. 19

Section 16	Page 92	#3
		#5 a,b [see comment below]
		#8

Comment on Page 92 Exercise 5: This is what the text is trying to say: Let $(X, \mathcal{T})$ be a topological space; let $(X, \mathcal{T}')$ be the same set with a possibly different topology. Similarly, let $(Y, \mathcal{U})$ and $(Y, \mathcal{U}')$ be a set Y with possibly different topologies $\mathcal{U}$ and $\mathcal{U}'$. Let us introduce the notation $\mathcal{T} \times \mathcal{U}$ to denote the product topology on $X \times Y$ gotten from the topologies $\mathcal{T}, \mathcal{U}$. In this notation, part(a) of the problem asks you to show that if $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{U} \subset \mathcal{U}'$ then $\mathcal{T}' \times \mathcal{U}'$ is finer than $\mathcal{T} \times \mathcal{U}$.

Section 17: Closed Sets and Limit Points

There are two ways to approach these ideas: One way is to define *closed* set as any set whose complement is open. The other way is to define closed set as any set which contains all of its limit–points. These lead to different definitions of *closure* of a set (definition page 95 vs. theorem 17.6). Fortunately, the two approaches ultimately agree (Corollary 17.7). There are reasons to prefer either approach; our text uses the first. The homework from this section that is due on Sept. 19 only uses the first idea (pages 93–96), and does not require *limit points* (pages 97-98). So you *can* do the problems based on reading just the first part of the section. However, if you want, you are welcome to use any theorems in the whole section when doing the homework. But please be careful when justifying various statement in your homework: For example,

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the complement of a closed set is open *by definition*; a closed set contains all of its limit points *by theorem* 17.6.

There may be more notes on this section posted Sunday. Meanwhile, here is the assignment.

Homework Due Wednesday Sept. 19

Section 17	Page 100	#2
		#3
		#6
		#8
		#9

Optional problem. Page 102 #21 b.

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