22M:132
Fall 07
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Handout 2
(Correction of Handout 1 plus continued discussion/HW)

Comments and Homework in Chapter 1

Chapter 1 contains material on sets, functions, relations, and cardinality that most (all?) of you have studied before. We will do a quick review of this material (Sections 1-7) just to give you a chance to [re-]master some of the details, practice writing careful proofs, learn the text's definitions that will be used throughout the course, and get [re-]minded of some foundational theorems that we will need during the course. We will skip Section 8, and discuss Sections 9-11 later when we need that material. I don't want us to spend to much time and energy worrying about details of set theory and logic. But there are some connections between topology and set theory/logic that should not be ignored. For example, the [topology] Tychonoff theorem on products of compact sets (Chapter 5) is logically equivalent to the Axiom of Choice.

Section 1: Sets. Unassigned practice problems: If you can work these quickly, that's good. If you have problems, get help from the TA.

Page 14	# 2 a,c,e,g,i,k,m,o,q
	# 3, 4

Section 2: Functions. Note definitions of *injective (1-1), surjective (onto), bijectiive (1-1 correspondence)*. Assigned HW problems give you a chance to practice with images and inverse-images of sets (very important for later that you have these ideas well-mastered) and various kinds of functions.

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Page 20	#1
	#2 f,g,h
	[It would be good also to practice all of #2,3,4,(and #5 if you feel ambitious)]

Section 3: Relations. Key ideas:

- Key ideas *equivalence relation* and *partition of a set.* Be able to prove the following, which is partly done in the text but not as a clearly stated theorem:

  Theorem. (1) *If $\sim$ is an equivalence relation on a set A, then the set of equivalence classes is a partition of A.*

  (2) *Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a partition of a set $\mathcal{A}$. Define a relation on $\mathcal{A}$ by $x \sim y \iff$ there exists $j \in J$ such that $x \in A_j$ and $y \in A_j$. Then $\sim$ is an equivalence relation.*

- Key idea *ordered set.* The usual relations of "<" and "≤" for numbers can be generalized to various kinds of orderings of sets sets. Find the text definitions of *partial order[ing]*, *strict partial order[ing]*, *linear order[ing].* The important distinctions are (a) whether each element of the set can be compared to each other element, and (b) whether an element can be related to itself. You may have to re-read the definitions each time you work on some theorems, but keep these main distinctions in mind and you should be fine.

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Page 28	#1
	#3 (Find the fallacy AND give a counterexample.)
	#4 (Probably will be easy for you - but this is a very important special case.)
	#11 (An exercise in being careful with the definitions.)
	#13 (May take some thought. Remember that "ordered" set means [page 24] there is a strict linear ordering.)

Section 4: The Integers and the Real Numbers. Just skim/browse this section. We will happily assume the existence and all the usual properties of $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. In particular, we will do "induction" proofs without worrying about any set-theory/logic technical complications.

Remark. *Just to keep us humble in thinking we understand these familiar sets: See the discussion in* http://en.wikipedia.org/wiki/Continuum_hypothesis *about the so-called "continuum hypothesis". We know $\mathbb{Z}_+$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same size as each other; and we know that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R} - \mathbb{Q}$ have the same size as each other, and are strictly larger than the sets in the first group; but we do not know (and, in a sense* **can not** *know) if there is a set whose size is properly in between those two.*

[End of handout. Discussion of Chapter 1 to be continued...]

Section 5: Cartesian products. Once you define an *indexed family of sets*, you can define unions, intersections, and products.

$$\bigcup_{j \in J} A_j = \{a \mid \exists j \in J \text{ such that } a \in A_j\}$$

$$\bigcap_{j \in J} A_j = \{a \mid \forall j \in J,\ a \in A_j\}$$

$$\prod_{j \in J} A_j = \{x : J \to \cup_{j \in J} A_j \mid \forall j \in J,\ x(j) \in A_j\}$$

If the index set J is $\{1, 2, \ldots, n\}$ or even all of $\mathbb{Z}_+$, we can write elements of the cartesian product as n-tuples or sequences, e.g.

$$(a_1, a_2, a_3) \text{ or } (a_1, a_2, a_3, \ldots) .$$

But if the index set is larger, e.g. all of $\mathbb{R}$, then we have to use the function notation to define and analyze the cartesian products..

We often will be dealing with cartesian products that involve crossing one set with itself some "number" of times. For example,

$$\mathbb{R}^3 \text{ or } \mathbb{R}^\omega \text{ or } \mathbb{R}^\mathbb{R} .$$

Another example that you have seen, or will see, in analysis courses is $\mathbb{R}^{[a,b]}$ and the subset

$$\mathcal{C}([a, b] = \{f : [a, b] \to R \mid f \text{ is continuous } \} .$$

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Page 38	#2 (might help to read # 1 first)
	#3d
	#4 d, e
(It would be good for you to	read/understand all of #1 – #4.)

Section 6. Finite sets. We will accept this material without fussing. You should read/skim to be sure you are comfortable with these ideas. In particular, a set is *finite* if it is (empty or) in 1-1 correspondence with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. A finite set cannot be put in 1-1 correspondence with a proper subset of itself. It is easy to see bijections between $\mathbb{Z}_+$ and proper subsets, so $\mathbb{Z}_+$ is not finite.

There is no assigned HW in Section 6.

Section 7. Countable and Uncountable sets. This section is very important. In addition to further honing your abilities to write proofs, the "facts" themselves will be used often in the course: Propositions such as:

- *a subset of a countable set is countable*, or
- *a countable union of countable sets is countable*, or
- *the cartesian product of n countable sets is countable*, or
- *any infinite product of sets is uncountable* (unless nearly all the sets are empty or singletons).

are things you should believe and understand, as well as being able to prove them.

There are two main ways to show that a set X is countable: One is to define explicitly a bijection between X and a set previously known to be countable; the other is to use some version of the following theorem.

Here is a slightly specialized version of text Theorem 7.1.

Theorem $(\widetilde{7.1})$**.**
a. *If* $X \subseteq \mathbb{Z}+$ *then* X *is countable.*
b. *If* X *is a set such that there exists a function* $f : Z_+ \to X$ *that is surjective, then* X *is countable.*

(Note: This proof in the text involves some set theory fussing that I would like to avoid. So for this theorem, you should understand the theorem well, but I won't ask you to reproduce the detailed proof.)

Proof. (Remark: Both parts of this proof use the fact that the usual size ordering of $\mathbb{Z}_+$ is a "well-ordering", i.e. a linear ordering such that each nonempty subset has a smallest element.)

Part (a). Define a bijection between X and some initial segment of $\mathbb{Z}_+$ (either $\{1, 2, \ldots, n\}$ or all of $\mathbb{Z}_+$) inductively as follows: Let x_1 denote the smallest element of X and define $f(x_1) = 1$. Let $X_1 = X \backslash \{x_1\}$. If X_1 is empty, we are done; if $X_1 \neq \emptyset$ then let $x_2 =$ the smallest element of X_1 and define $f(x_2) = 2$. Continue "inductively" [that is where the set theory fussing happens] to get the desired bijection f between X and a set $\{1, 2, \ldots, n\}$ or all of $\mathbb{Z}_+$.

Part(b). For each $x \in X$, let A_x be the pre–image $f^{-1}(x)$. The sets $\{A_x\}_{x \in X}$ are a partition of $\mathbb{Z}_+$. We know (page 28, Exercise 4) that this set of equivalence classes is in $1 - 1$ correspondence with X. Form a set $W \subseteq \mathbb{Z}_+$ by picking one element from each set A_x. (For example, again using the well-ordering of Z_+, we can pick a_x to be the smallest element of A_x.) Because the sets A_x are pairwise disjoint, the elements a_x are all distinct from each other. So the set $W = \{a_x\}_{x \in X}$ is in $1 - 1$ correspondence with X. But W is a subset of $\mathbb{Z}_+$ so W is countable, and therefore X is countable.

□

Corollary. *Theorem 7.1 as stated in the text.*

Proof. "Exercise for the reader". Use $\widetilde{7.1}$. □

Corollary (7.4)**.** $\mathbb{Z}_+ \times \mathbb{Z}_+$ *is bijectively equivalent to* $\mathbb{Z}_+$*.*

Proof. (Note: This illustrates the second main technique for proving that sets are countable: Exhibit a bijection that does the job.) The function $f : (n, m) \to 2^n\, 3^m$ is a bijection between $\mathbb{Z}_+ \times \mathbb{Z}_+$ and a subset B of $\mathbb{Z}_+$. □

Corollary (call it 7.4.1)**.** *If* X, Y *are countable sets, then* $X \times Y$ *is countable.*

Proof. "Exercise for the reader". This follows from Corollary 7.4 and Theorem 7.1. □

Corollary. *A finite product of countable sets is countable.*

Proof. "Exercise for the reader". This follows by induction (and your earlier homework page 38 #2a) from the previous Corollary. □

Theorem (7.5)**.** *A countable union of countable sets is countable.*

Proof. (Slightly different from text's proof.) The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a countably infinite union of copies of $\mathbb{Z}_+$, so any other countable union of countable sets can be injectively mapped into $\mathbb{Z}_+ \times \mathbb{Z}_+$. (Roughly, use the first $\mathbb{Z}_+$ coordinate to represent the index and the second $\mathbb{Z}_+$ factor to capture the set elements.). So any countable union of countable sets is bijectively equivalent to a subset of the countable set $\mathbb{Z}_+ \times \mathbb{Z}_+$. □

Theorem. $\mathbb{Q}$ *is countable.*

Proof. Let's try to do this in careful steps.

(1) $\mathbb{Z}$ is countable. Exhibit a specific bijection between $\mathbb{Z}$ and $\mathbb{Z}_+$.
(2) $\mathbb{Z}_* = \mathbb{Z}\backslash\{0\}$ is countable. This follows from step (1) and Theorem (7.4).
(3) $\mathbb{Z} \times \mathbb{Z}_*$ is countable. This follows from steps (1), (2), and Corollary (7.4.1) above.
(4) The function $(m, n) \to \frac{m}{n}$ is a surjection of $\mathbb{Z} \times \mathbb{Z}_*$ onto $\mathbb{Q}$. Now invoke Theorem (7.4) again.

□

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Page 51	#3
	#4
	#5 c, d, e, f, g, h
(You can "justify your answers" by	citing theorems or earlier problems.)

If you want some more problems to think about, try showing

- There is a bijection between $\mathbb{R}$ and $\mathbb{R}_* = \mathbb{R}\backslash\{0\}$.
- There is a bijection between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$.

[END OF HANDOUT ON CHAPTER 1]