22M:132
Fall 07
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Handout 3

More Discussion of Section 17

limit point · closed set · closure · interior · boundary

This section introduces several ideas and words (the five above) that are among the most important and widely used in our course and in many areas of mathematics. The other "universally important" concepts are *continuous (Sec. 18), homeomorphism* (Sec. 18), *connected* (Sec. 23) and *compact* (Sec. 26).

We are going to go through the section with most definitions and theorems in the text order. However, I want you to have a sense of the flow and relations between the ideas before we get into the details.

Closure, Limit Point, and Interior

As mentioned earlier, there are two ways to define *closed* set. One way uses the idea of *limit points*, one doesn't. Here are the two definitions, given as a theorem saying that the two are equivalent.

Theorem (Definition page 93 + Corollary 17.7)**.** *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then*

$$X - A \textit{ is open } \iff A \textit{ contains all of its limit points.}$$

Similarly, there are two ways to define the *closure* of a set, and we can prove a theorem saying they are equivalent.

Definition (text def., page 95)**.** *For a set $A \subseteq X$, the* closure *of A is the intersection of all closed sets in X that contain A. That is,*

$$\textit{Closure of } A = \bigcap \{C \mid A \subseteq C \textit{ and } C \textit{ is a closed set in } X\}$$

Definition (alternate def., proved equivalent by Theorem 17.6)**.** *For a set $A \subseteq X$, the* closure *of A is the union of A with all its limit points.*

We write $\bar{A}$ to denote the closure of set A.

Now, what are the *limit points*? Once again, there are two definitions, but they are not equivalent: one is simple (but not what people use), the other is a bit more complicated (but makes a distinction that people have found useful enough to maintain).

Suppose $(X, \mathcal{T})$ is a topological space, $A \subseteq X$, and $x \in X$. We ask the question, *"How close is x to A?"*.

We use the open sets that comprise $\mathcal{T}$ to measure *how close*. To make the math and our intuition more similar, let us say that a set $U \subseteq X$ is a *neighborhood* of x if $x \in U \in \mathcal{T}$.

There are three possible situations:

(1) $x \in A$
(2) $x \notin A$ but each neighborhood of x intersects A
(3) there is some neighborhood of x disjoint from A.

Definition. *In situation (2), we say x is a* limit point *of A.*

If we just said, "Each neighborhood of x intersects A", that would describe situations (1) and (2) without distinguishing between them; that would be the simpler definition of "limit point" that we do not adopt. Note that in situation (3), neither (1) nor (2) can occur. But (1) and (2) are logically independent from each other: either can happen with, or without, the other. *[Think of various topologies that we have discussed and find examples of the various situations.]*

We do have a name for the situation where each neighborhood of x hits A:

Theorem (17.5)**.** $x \in \bar{A} \iff$ *each neighborhood of x hits A.*

CAUTION (mathematics)**.** *A set is closed if and only if its complement is open. The ideas of* closed *and* open *are "dual" to each other; any time you have a theorem about open sets, you also have a theorem about closed sets – but the properties are* **not** *negations of each other. It is quite possible for a set to be simultaneously closed and open.*

Example. *Let $X \subset \mathbb{R}$ be the set $[0,1] \cup \{2\}$ with the subspace topology. The set $A = \{2\}$ is both closed and open in X. Also the set $[0,1]$ is both closed and open in X.*

CAUTION (vocabulary)**.** *There are some authors who use the ugly word "clopen". I* **hate** *the word "clopen". Please do not use that word.*

Definition. *The* **interior** *of a set A is not as tricky to define or understand as the closure.*

$$int(A) = \bigcup \{U \mid U \text{ is open and} U \subseteq U\}$$

Here is a sometimes useful way to think about *interior* and *closure*:

- The interior of A is the largest open set inside A.
- The closure of A is the smallest closed set containing A.

Remark (Notation)**.** *The text uses "int A" to denote the interior of set A. Some other authors use a small circle (like a "degree" mark) above the set for this; so A° is another way to write int(A). [And sometimes we write ""int A" and sometimes "int(A)"; let context and your own taste be your guides.]*

Optional problem. Suppose we are in some topological space, start with a set A and generate a sequence of sets as follows:

$$\text{int}(A)\ ,\ \text{closure}(\text{int}(A))\ ,\ \text{int}(\text{closure}(\text{int}(A)))\ ,\ \text{closure}(\text{int}(\ldots))))\ ,\ \text{etc.}$$

Must this sequence always be finite? Is there a universal upper bound to the number of different sets that can arise this way?

Boundary of a set

(This is introduced in Problem 19, page 102. You should view Problems 19 & 20 as additional sections of the text to study.)

For this discussion, think in terms of trying to approximate (i.e. get arbitrarily close to) a point x using points in a set A. The closure of A is all the points that can be approximated from within A: either the point can be "approximated" perfectly because it already is in A, or there are points of A that are as close as we want to x.

The closure of the complement, $\overline{X - A}$, is all the points that can be approximated from outside A.

The points that can be approximated from within A and from within $X - A$ are called the *boundary* of A:

$$\text{bd}A = \overline{A} \cap \overline{X - A}\ .$$

There are many theorems relating these "anatomical features" (interior, closure, limit points, boundary) of a set. While I do want you to know some of the relations, the main point of all these homework exercises is to get you familiar with the ideas and how to work with them, so that in any given situation, you can cook up a proof or counterexample as needed.

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Hausdorff property

The ideas of "open" and "closed" sets are universal to all topological spaces. But now we encounter a *property* of a topology where some topologies have the property and others don't.

If (as we do) think of neighborhoods of a point as being sets of points that are, in some sense, close to the given point, then it might be annoying, or trivial, or somehow degenerate, to have two points that have exactly the same neighborhoods. On the real line $\mathbb{R}$, given any two points, we can find neighborhoods of the points that are disjoint from each other. *This property is why a sequence of numbers cannot converge to two different limits.*

Definition. *We say a space $(X, \mathcal{T})$ has the Hausdorff property if $\forall x, y \in X$, if $x \neq y$ then there exist open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

In a "Hausdorff space" (i.e. a space with the Hausdorff property), each singleton set $\{x\}$ is closed; each finite set of points is closed; a convergent sequence has only one limit.

Another way to state the Hausdorff property is to say that *we can separate points from each other* with open sets. The ability to "separate" points (or various kinds of sets) from each other by finding disjoint neighborhoods is sometimes an important property of a topology. For example, so-called Urysohn's lemma depends on being able to separate disjoint closed sets with disjoint neighborhoods; this leads to the Tietze extension theorem. Both of these say that given suitable conditions, we can construct a continuous function having certain desired properties. You will see these topics later in the course. One monster theorem is the Urysohn metrization theorem (section 34): If $(X, \mathcal{T})$ has property T_4 below, and the topology has a countable basis, then there exists a metric d on X such that the metric topology is precisely $\mathcal{T}$. This is heady stuff for later in the course. Meanwhile, I hope you agree that being able to say, "THE limit of the sequence (a_n)" as opposed to "ONE OF THE limits of the sequence (a_n) could be important.

In the misty history of topology, some people tried to organize the various kinds of "separation properties" in a hierarchy. The German word for the verb "to separate" is *trennen* and this why the original list of separation properties (see page 211) had labels with letter "T".

(T_0) Given two distinct points x, y, there exists a neighborhood of one point that misses the other.

(T_1) (see page 99) Given two distinct points x, y, each point has a neighborhood that misses the other point. (The text defines the T_1 property as knowing that all finite sets are closed. In one of the homework problems, you show that is equivalent to what I have written as the definition.)

(T_2) Given two distinct points x, y, there exist disjoint neighborhoods of x and y.

(T_3') Given a point and a closed set not containing the point, there exist disjoint neighborhoods.

(T_4') Given two disjoint closed sets, there exist disjoint neighborhoods.

There are more T-properties we will see later. Right now, the only one you have to worry about is the Hausdorff property (T_2) and one exercise involving (T_1).

If you do not assume that finite sets are closed (i.e the T_1 property) then (T_3') and (T_4') are not stronger than the lower numbered properties. Notice I stuck in a "prime" after the labels. Our text defines T_3 and T_4 to include T_1. That way $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. One can get tangled up in all this $T-$stuff; I don't want us to get hung up on that, but later in the course we will work with these higher separation properties.

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[end of handout]