Problem 1.

Here are four topologies on the set $\mathbb{R}$. For each pair of topologies, determine whether one is a refinement of (i.e. contains) the other. [Justify your claims.]

[Note: So you have $\binom{4}{2} = 6$ comparisons to make.]

A. The usual (i.e. standard) topology.
B. The discrete topology.
C. The lower-limit topology (recall $\mathbb{R}$ with this the topology is denoted $\mathbb{R}_l$).
D. The counter-finite (i.e. finite-complements) topology.

Answers:

a. Counter-finite is strictly coarser than Standard.
   Proof: Each finite set in $\mathbb{R}$ is closed in the standard topology, so each set whose complement is finite is open in the standard topology. However, an open interval $(1, 2)$ is open in the standard topology; but its complement is infinite, so the interval $(1, 2)$ is not open in the finite-complements topology.

b. Standard is strictly coarser than lower-limit.
   Proof: First show that each basis element for the standard topology is open in the lower-limit topology: $(a, b) = \bigcup \{[x, b) \mid b > x > a\}$. On the other hand, a basis set $[a, b)$ for the lower limit cannot be a union of basis sets for the standard topology since any open interval in $\mathbb{R}$ containing point $a$ must contain numbers less than $a$.

c. Lower-limit is strictly coarser than Discrete.
   Proof: In the Discrete topology, every set is open; so the lower-limit topology is coarser-than-or-equal-to the Discrete topology. On the other hand, the singleton set $\{0\}$ is open in the discrete topology but is not a union of half-open intervals.

The remaining comparisons follow (by transitivity) from the three above; the four topologies are linearly ordered by proper inclusion.
Problem 2.

Let \((X, \mathcal{T})\) be a topological space, and let \(\mathcal{S}\) be a sub-basis for \(\mathcal{T}\).

Prove: If \(\mathcal{S}\) is countable, then there exists a countable basis for \(\mathcal{T}\).

**Proof.** The set of all finite intersections of elements of \(\mathcal{S}\) is a basis \(\mathcal{B}\) for \(\mathcal{T}\). We shall define a countable set and a function from that countable set onto \(\mathcal{B}\); we have a theorem that the image of a countable set is countable, so we will be able to conclude that \(\mathcal{B}\) is countable.

For each \(n \in \mathbb{Z}_+\), let \(\mathcal{S}_n = (\mathcal{S} \times \mathcal{S} \times \ldots \times \mathcal{S})\) \(n\) times. A finite cartesian product of countable sets is countable, so each \(\mathcal{S}_n\) is countable. Let \(\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n\). A countable union of countable sets is countable, so \(\mathcal{S}\) is countable.

Now define a surjective function \(\phi : \mathcal{S} \to \mathcal{B}\) by
\[
\phi(S_1, S_2, \ldots S_k) = S_1 \cap S_2 \cap \ldots S_k.
\]

Problem 3.

Let \((X, \mathcal{T})\) be a topological space, with \(A, B \subseteq X\).

Prove:

\[
\overline{A \cup B} = \overline{A} \cup \overline{B}.
\]

**Proof.** There are various ways to prove this; here is one.

We have the following theorems:

- The closure of a set is closed.
- The union of two closed sets is closed.
- The closure of a set is defined to be the intersection of all closed sets containing the given set.
- and from the last statement above, if \(P \subseteq Q\) then \(\overline{P} \subseteq \overline{Q}\).

So we can argue as follows:

\[
A \subseteq \overline{A}, \text{ and } B \subseteq \overline{B} \implies A \cup B \subseteq \overline{A} \cup \overline{B}.
\]

Since \(A \cup B \subseteq \overline{A} \cup \overline{B}\), the closure of \(A \cup B\) must be contained in that closed set; i.e.

\[
\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.
\]

Conversely,

\[
A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B}
\]

and likewise for \(\overline{B}\).
Problem 4.

Let \( \{X_\alpha\}_{\alpha \in J} \) be a family of nonempty topological spaces; give \( \Pi_{\alpha \in J}X_\alpha \) the product topology.

Prove:
If \( \Pi_{\alpha \in J}X_\alpha \) has the Hausdorff property, then each \( X_\alpha \) has the Hausdorff property.

Proof. (For notational simplicity, we will write the product as if the index set is assumed to be countable, \( \Pi_nX_n \) where \( n \) runs from 1 to some \( N \) or \( \infty \).)

Fix an index \( k \) and show that the factor space \( X_k \) is Hausdorff. Let \( x, y \) be points in \( X_k \) with \( x \neq y \). Since the factor spaces are all nonempty [this is where we need that hypothesis], in each space \( X_n \) (\( n \neq k \)) pick a point \( t_n \). Let \( \hat{x} \in \Pi_nX_n \) be the point that is \( x \) in coordinate \( k \) and \( t_n \) in each other coordinate \( n \). Similarly, let \( \hat{y} \in \Pi_nX_n \) be the point that is \( y \) in coordinate \( k \) and \( t_n \) in each other coordinate \( n \). Since the product is Hausdorff, there exist disjoint open neighborhoods \( U(\hat{x}) \) and \( V(\hat{y}) \). Since we are using the product topology [this argument works equally well with the “box” topology], inside \( U, V \) there are basic neighborhoods of \( \hat{x} \) and \( \hat{y} \); since \( U, V \) are disjoint, these subsets are disjoint. So we have disjoint product neighborhoods of \( \hat{x}, \hat{y} \):

\[
U' = U_1 \times U_2 \times \ldots \quad \text{and} \quad V' = V_1 \times V_2 \times \ldots
\]

Note that \( x \in U_k \) and \( y \in V_k \), and for each \( n \neq k \), \( t_n \in U_n \cap V_n \). If there exists a point \( z_k \in X_k \) such that \( z_k \in U_k \cap V_k \) then we have a point in \( U' \cap V' \), namely let \( \hat{z} \) be \( z_k \) in coordinate \( k \) and \( t_n \) in each coordinate \( n \neq k \). Thus \( U_k \) and \( V_k \) are disjoint neighborhoods of \( x, y \) in \( X_k \).

We started with an arbitrary \( X_k \) and any two points of \( X_k \) and found disjoint neighborhoods in \( X_k \). Thus each \( X_k \) is Hausdorff. \( \square \)

Problem 5.

Let \( Y \) be an ordered set in the order topology. Let \( X \) be a topological space and let \( f, g : X \to Y \) be continuous functions.

a. Show that the set \( A = \{x \in X \mid f(x) \leq g(x)\} \) is closed in \( X \).

b. Let \( h : X \to Y \) be the function

\[
h(x) = \min \{f(x), g(x)\}.
\]

Show the function \( h \) is continuous.

[Hints: (a) Show \( X - A \) is open. (b) Use the "pasting lemma".]
Proof. Let \( B = \{ x \in X \mid f(x) > g(x) \} \). We will show \( B \) is open.

All open intervals \((a, b)\) and open rays \((a, \infty), (-\infty, b)\) are open sets in the order topology on \( Y \), and these sets form a basis; the rays form a sub-basis.

Let \( x \) be any point in \( B \). We want to find a neighborhood of \( x \) contained in \( B \). So we need to find an open neighborhood \( U(x) \) such that for each \( w \in U \), \( f(w) > g(w) \).

As in the proof that an order topology has the Hausdorff property, we claim there exist disjoint sub-basic neighborhoods [i.e. open rays] \( U \) of \( f(x) \) and \( V \) of \( g(y) \). [If there exists \( w \in (g(y), f(x)) \) then use \((-\infty, w)\) and \((w, \infty)\). If the open interval \((g(y), f(x))\) is empty, then use the rays \((-\infty, f(x))\) and \((g(y), \infty)\).] Then \( f^{-1}(U) \cap g^{-1}(V) \) is a neighborhood of \( x \) on which \( f \) is strictly larger than \( g \).

For part (b), note that the function \( h \) is just \( f \) on the closed set \( A \) and \( g \) on the dual closed set \( C \) where \( g(x) \leq f(x) \). On the intersection \( A \cap C \), we have \( f(x) = g(x) \). So by the “pasting lemma”, this function is well-defined and continuous.

\[ \square \]

**Problem 6.**

Suppose \( X,Y \) are topological spaces, and \( f : X \to Y \) is a continuous function. In the space \( X \times Y \) (with the product topology) we define a subspace \( G \) called the “graph of \( f \)” as follows:

\[ G = \{(x, y) \in X \times Y \mid y = f(x)\} \, . \]

Prove: \( G \) is homeomorphic to \( X \).

(Be sure to state clearly whatever theorem(s) you use in doing this proof.)

**Proof.** Define \( \phi : X \to G \) by \( \phi(x) = (x, f(x)) \). As the cartesian product of two continuous functions, \( \phi \) is continuous. Check that \( \phi \) is a bijection [1-1 is because \( f \) is a function; surjective is by definition of the “graph of \( f \)”.] The function \( \phi^{-1} \) is just the restriction to \( G \) of the projection map \( \pi_X : X \times Y \to X \). We have a previous theorem that \( \pi_X \) is continuous and a theorem that the restriction of a continuous function to a subspace is continuous. So \( \phi^{-1} : G \to X \) is continuous. \[ \square \]