22M:132
Fall 07
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Exam 1 Solutions

Problem 1.

Here are four topologies on the set $\mathbb{R}$. For each pair of topologies, determine whether one is a refinement of (i.e. contains) the other. [Justify your claims.]

[note: So you have $\binom{4}{2} = 6$ comparisons to make.]

A *The usual (i.e. standard) topology.*
B *The discrete topology.*
C *The lower-limit topology (recall R with this the topology is denoted $\mathbb{R}_\ell$).*
D *The counter-finite (i.e. finite-complements) topology.*

Answers:

a *Counter-finite is strictly coarser than Standard.*
Proof: Each finite set in $\mathbb{R}$ is closed in the standard topology, so each set whose complement is finite is open in the standard topology. However, an open interval $(1, 2)$ is open in the standard topology; but its complement is infinite, so the interval $(1, 2)$ is not open in the finite-complements topology.

b *Standard is strictly coarser than lower-limit.*
Proof: First show that each basis element for the standard topology is open in the lower-limit topology: $(a, b) = \cup\{[x, b) \mid b > x > a\}$. On the other hand, a basis set $[a, b)$ for the lower limit cannot be a union of basis sets for the Standard topology since any open interval in $\mathbb{R}$ containing point a must contain numbers less than a.

c *Lower-limit is strictly coarser than Discrete.*
Proof: In the Discrete topology, every set is open; so the Lower-limit topology is coarser-than-or-equal-to the Discrete topology. On the other hand, the singleton set $\{0\}$ is open in the discrete topology but is not a union of half-open intervals.

The remaining comparisons follow (by transitivity) from the three above; the four topologies are linearly ordered by proper inclusion.

Problem 2.

Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S}$ be a sub-basis for $\mathcal{T}$.

Prove: If $\mathcal{S}$ is countable, then there exists a countable basis for $\mathcal{T}$.

Proof. The set of all finite intersections of elements of $\mathcal{S}$ is a basis $\mathcal{B}$ for $\mathcal{T}$. We shall define a countable set and a function from that countable set onto $\mathcal{B}$; we have a theorem that the image of a countable set is countable, so we will be able to conclude that $\mathcal{B}$ is countable.
For each $n \in \mathbb{Z}_+$, let $\mathcal{S}_n = (\mathcal{S} \times \mathcal{S} \times \ldots \times \mathcal{S})$ n times. A finite cartesian product of countable sets is countable, so each $\mathcal{S}_n$ is countable. Let $\mathbb{S} = \cup_{n=1}^{\infty} \mathcal{S}_n$. A countable union of countable sets is countable, so $\mathbb{S}$ is countable.
Now define a surjective function $\phi : \mathcal{S} \to \mathcal{B}$ by

$$\phi(S_1, S_2, \ldots S_k) = S_1 \cap S_2 \cap \ldots S_k \; .$$

□

Problem 3.

Let $(X, \mathcal{T})$ be a topological space, with $A, B \subseteq X$.
Prove:

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \; .$$

Proof. There are various ways to prove this; here is one.

We have the following theorems:

- The closure of a set is closed.
- The union of two closed sets is closed.
- The closure of a set is defined to be the intersection of all closed sets containing the given set.
- and from the last statement above, if $P \subseteq Q$ then $\overline{P} \subseteq \overline{Q}$.

So we can argue as follows:

$$A \subseteq \overline{A} \; , \text{ and } B \subseteq \overline{B} \implies A \cup B \subseteq \overline{A} \cup \overline{B} \; .$$

Since $A \cup B \subseteq$ the closed set $\overline{A} \cup \overline{B}$, the closure of $A \cup B$ must be contained in that closed set; i.e.

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \; .$$

Conversely,

$$A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B}$$

and likewise for $\overline{B}$.

□

Problem 4.

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces; give $\Pi_{\alpha\in J} X_\alpha$ the product topology.

Prove:
If $\Pi_{\alpha\in J} X_\alpha$ has the Hausdorff property, then each X_α has the Hausdorff property.

Proof. (For notational simplicity, we will write the product as if the index set is assumed to be countable, $\Pi\, X_n$ where n runs from 1 to some N or ∞.)
Fix an index k and show that the factor space X_k is Hausdorff. Let x, y be points in X_k with $x \neq y$. Since the factor spaces are all nonempty [this is where we need that hypothesis], in each space X_n $(n \neq k)$ pick a point t_n. Let $\hat{x} \in \Pi_n X_n$ be the point that is x in coordinate k and t_n in each other coordinate n. Similarly, let $\hat{y} \in \Pi_n X_n$ be the point that is y in coordinate k and t_n in each other coordinate n. Since the product is Hausdorff, there exist disjoint open neighborhoods $U(\hat{x})$ and $V(\hat{y})$. Since we are using the product topology [this argument works equally well with the "box" topology], inside U, V there are basic neighborhoods of $\hat{x}$ and $\hat{y}$; since U, V are disjoint, these subsets are disjoint. So we have disjoint product neighborhoods of $\hat{x}, \hat{y}$:

$$U' = U_1 \times U_2 \times \ldots \;\; and \;\; V' = V_1 \times V_2 \times \ldots$$

Note that $x \in U_k$ and $y \in V_k$, and for each $n \neq k$, $t_n \in U_n \cap V_n$. If there exists a point $z_k \in X_k$ such that $z_k \in U_k \cap V_k$ then we have a point in $U' \cap V'$, namely let $\hat{z}$ be z_k in coordinate k and t_n in each coordinate $n \neq k$. Thus U_k and V_k are disjoint neighborhoods of x, y in X_k.
We started with an arbitrary X_k and any two points of X_k and found disjoint neighborhoods in X_k. Thus each X_k is Hausdorff.

□

Problem 5.

Let Y be an ordered set in the order topology. Let X be a topological space and let $f, g : X \to Y$ be continuous functions.

a . *Show that the set $A = \{x \in X \mid f(x) \leq g(x)\}$ is closed in X.*

b . *Let $h : X \to Y$ be the function*

$$h(x) = min\,\{f(x), g(x)\}\,.$$

Show the function h is continuous.

[Hints: (a) Show $X - A$ is open. (b) Use the "pasting lemma".]

Proof. Let $B = \{x \in X \mid f(x) > g(x)\ \}$. We will show B is open.
All open intervals (a, b) and open rays (a, ∞), $(-\infty, b)$ are open sets in the order topology on Y, and these sets form a basis; the rays form a sub-basis.
Let x be any point in B. We want to find a neighborhood of x contained in B. So we need to find an open neighborhood $U(x)$ such that for each $w \in U$, $f(w) > g(w)$.
As in the proof that an order topology has the Hausdorff property, we claim there exist disjoint sub-basic neighborhoods [i.e. open rays] U of $f(x)$ and V of $g(y)$. [If there exists $w \in (g(y), f(x))$ then use $(-\infty, w)$ and (w, ∞). If the open interval $(g(y), f(x)$ is empty, then use the rays $(-\infty, f(x))$ and $(g(y), \infty)$.] Then $f^{-1}(U) \cap g^{-1}(V)$ is a neighborhood of x on which f is strictly larger than g.

For part (b), note that the function h is just f on the closed set A and g on the dual closed set C where $g(x) \leq f(x)$. On the intersection $A \cap C$, we have $f(x) = g(x)$. So by the "pasting lemma", this function is well-defined and continuous.

□

Problem 6.

Suppose X, Y are topological spaces, and $f : X \to Y$ is a continuous function. In the space $X \times Y$ (with the product topology) we define a subspace G called the "graph of f" as follows:

$$G = \{(x, y) \in X \times Y \mid y = f(x)\}\ .$$

Prove: G is homeomorphic to X.

(Be sure to state clearly whatever theorem(s) you use in doing this proof.)

Proof. Define $\phi : X \to G$ by $\phi(x) = (x, f(x))$. As the cartesian product of two continuous functions, ϕ is continuous. Check that ϕ iss a bijection [1-1 is because f is a function; surjective is by definition of the "graph of f".] The function ϕ^{-1} is just the restriction to G of the projection map $\pi_X : X \times Y \to X$. We have a previous theorem that π_X is continuous and a theorem that the restriction of a continuous function to a subspace is continuous. So $\phi^{-1} : G \to X$ is continuous. □