22M:132
Fall 07
J. Simon

Notes and Homework on

Locally Compact Spaces

Compact spaces (especially compact Hausdorff spaces) are extremely "nice" - as we have already studied (optimization problems have solutions; continuous functions are uniformly continuous; integrals exist). There is a more general class of spaces that are important (for example, they include $\mathbb{R}^n$) and that arise a lot in analysis (see, for example, the "Riesz representation theorem"). These spaces are too big to be compact, but they are compact when looked at from close-up. More precisely,...

Definition. *A space X is* locally compact *if for each $x \in X$, there exists an open neighborhood U of x with closure $\bar{U}$ compact.*

When X is also Hausdorff, the property of local compactness becomes much stronger. Let's state this as a theorem.

Theorem 1. *If X is locally compact and Hausdorff, $x \in X$, and U is any neighborhood of x, then there exists a neighborhood V of x such that the closure $\bar{V}$ is compact and $\bar{V} \subseteq U$.*

Remark. *So not only does x have <u>some</u> neighborhood with compact closure, it has many; in fact, it has arbitrarily small neighborhoods with compact closure.*

The text proves this theorem by first embedding X in its "one-point compactificaton". Instead, let's prove the theorem more directly, and then use this tool to help understand the one-point compactification space. Ultimately, we are all doing the same "dirty work", just changing the order in which we encounter various issues. (And I think the approach in these notes makes the issues clearer.)

Lemma 1.1. *If X is Hausdorff, $x \in X$, and C is a compact subset of X with $x \notin C$, then there exist disjoint neighborhoods $U(x)$ and $V(C)$.*

Proof. This is stated as Lemma (26.4) in the text. The technique for this proof is something you should know well, useful for other theorems, so here is the proof.

Since X is Hausdorff, for each point $y \in C$, there are disjoint neighborhoods of x and y; let's call these $U_y(x)$ and $V_y(y)$. The set C is covered by $\{V_y \;:\; y \in C\}$ and, since C is compact, there is a finite subcover $\{V_{y1}, \ldots, V_{yn}\}$. So $U = U_{y1} \cap \ldots \cap U_{yn}$ and $V = V_{y1} \cup \ldots \cup V_{yn}$ are disjoint neighborhoods of x and C respectively.

□

Lemma 1.2. *In a Hausdorff space X, suppose U is a neighborhood of a point x and $bd\,U$ is compact. Then there exists a neighborhood V of x such that the closure $\bar{V} \subseteq U$.*

Proof. By assumption, $bd\,U$ is compact. Then, by Lemma (1.1), there exist disjoint neighborhoods W of x and W' of $bd\,U$. Note this implies that the closure $\bar{W}$ is disjoint from $bd\,U$. Let $V = U \cap W$. Then

$$\bar{V} \subseteq \bar{U} \cap \bar{W} = (U \cup bd\,U) \cap \bar{W} = (U \cap \bar{W}) \cup (bd\,U \cap \bar{W}) = (U \cap \bar{W}) \cup \emptyset \subseteq U\ .$$

□

Remark. *The idea in the preceding lemma is that if we can separate x from the boundary of a neighborhood $U(x)$ then we can shrink U to a neighborhood that is "deep" within U, that is the closure of the new neighborhood is contained in U.*

Proof of Theorem 1. We have $x \in U$, where U is a given neighborhood of x. By definition of local compactness, there exists a[nother] neighborhood W of x such that the closure $\bar{W}$ is compact. This makes any closed set contained in $\bar{W}$ also compact.

Consider the set $V_1 = U \cap W$. We might hope that V_1 is the desired neighborhood of x; it certainly is contained in U. But its closure is, in general, not contained in U,. So we have to "trim it down" a little.

The set $bd\,V_1$ is closed and contained in $\bar{V_1} \subseteq \bar{W}$ which is compact, so $bd\,V_1$ is compact. By Lemma 1.2, there exists a neighborhood $V(x)$ such that the closure $\bar{V} \subseteq V_1$; but since $V_1 = U \cap W$, this says $\bar{V} \subseteq U$.

□

Remark (for the future)**.** *Along with finding neighborhoods of a point that lie deep within a given one, we also can use the same kind of thinking (separate points from compact sets, or separate compact sets from each other) to get large families of nested neighborhoods. In fact, we can construct inductively a countable family of neighborhoods of x inside a given U where the countable family is indexed by rationals of the form $\frac{j}{2^n}$ for all positive integers j and n, such that the containment relations between the neighborhoods is the same as for the intervals $[0, \frac{j}{2^n}]$. This ultimately lets us construct continuous functions from X to $\mathbb{R}$ that "separate points" or "separate points from closed sets". In a [locally] compact Hausdorff space, given two points A, B or a point A and a closed set B missing A, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$. This property is sometimes called* completely regular. *We'll see more theorems like this in later sections.*