22M:132
Fall 07
J. Simon

# The One-Point Compactification

(Sec. 29 contd.)

Suppose we start with the closed unit interval $[0, 1]$ (call it $\hat{X}$) and remove the two endpoints, leaving the open interval $(0, 1)$, call that new space $X$. Now imagine someone meeting space $X$ for the first time. Can they somehow "know" that $X$ used to live inside a compact space? And that $X$ lived in the compact space in a special way, namely the whole compact space was just the closure of [its subset] $X$? The answer is "yes, but...". There are different ways to recover the compact space $\hat{X}$ from $X$. And different approaches may give different spaces $\hat{X}$.

For example, we started with $\hat{X} = [0, 1]$ and deleted two endpoints to get the subset $X$. But another person could have started with the circle $S^1$ and removed just one point to get a subspace homeomorphic to our $X$. So we have at least two different ways to *compactify* $(0, 1)$: Adjoin one new point to obtain $\hat{X} \cong S^1$ or adjoin two new points to obtain $\hat{X} \cong I^1$.

Here is another example: Suppose $\hat{Y}$ is a compact space obtained by joining three arcs together at one common endpoint [so the space looks like a letter "Y"]. Remove all three of the exposed endpoints to obtain the noncompact space $Y$. We can compactify $Y$ in at least two different ways:

1. Add one new point and tie all three exposed ends of $Y$ to that point. The space $\hat{Y}$ obtained this way looks like a Greek letter $\Theta$, topologically a circle plus one diameter.
2. Add three new points to recover exactly the original space $\hat{Y}$ that we started with.
3. Adjoin two new points, one to the end of one exposed arc, and the other to become the common endpoint of the two other arcs. This compactification looks like a circle with an arc attached, topologically equivalent to a letter P.

People have studied many possible ways to compactify noncompact spaces. One might focus on how many different ways are there to "get to infinity" (theory of "ends" and "end point compactification". Or focus on getting the compactification to be a manifold ("missing-boundary manifolds"). Or try to get a compactification that allows us to extend functions $f : X \to C$ to functions $\hat{f} : \hat{X} \to C$ ("Stone-Čech compactification, text Section 38)

To give a taste of this topic, in text Sec. 29, we study the (Alexandroff) one-point compactification. Add one point to $X$ called $\infty$.

**Definition.** *Suppose $X$ is a topological space, with topology $\mathcal{T}$. Let $\infty$ denote some abstract point that is not in $X$ and let $\hat{X}$ be the set $X \cup \{\infty\}$.*

Define a topology $\hat{\mathcal{T}}$ on $\hat{X}$ as follows:

(1) Each open set in $X$ is included in $\hat{\mathcal{T}}$, that is $\mathcal{T} \subseteq \hat{\mathcal{T}}$.
(2) For each compact set $C \subseteq X$, define an element $U_C \in \hat{\mathcal{T}}$ by $U_C = (X - C) \cup \{\infty\}$.

**Theorem.** *If $X$ is a locally compact Hausdorff space, then the space $\hat{X} = X \cup \{\infty\}$ satisfies:*

(1) *$\hat{\mathcal{T}}$ is a topology on $\hat{X}$.*
(2) *$(\hat{X}, \hat{\mathcal{T}})$ is a compact Hausdorff space.*
(3) *If $X$ is not itself compact, then $X$ is dense in $\hat{X}$. $X$ always is open in $\hat{X}$.*

*Proof.* (1) Show $\hat{\mathcal{T}}$ is a topology.

You can/should write out details for this. The key facts are

a. Since $X$ is Hausdorff, the sets $X - C$ are open in $X$.
b. A finite union $C_1 \cup \ldots \cup C_n$ of compact sets in $X$ is compact.
c. A closed subset of a compact set $C$ in $X$ is also compact.

(2) Show $(\hat{X}, \hat{\mathcal{T}})$ is compact.

First, consider any open cover of $\hat{X}$. One of the sets contains $\infty$. So that set is of the form $(X - C) \cup \{\infty\}$ for some compact subset of $X$. Then we need only finitely many more of the open cover sets to cover $C$ and thus have a finite subcover for $\hat{X}$.

(3) Show $(\hat{X}, \hat{\mathcal{T}})$ is Hausdorff. Let $x, y \in \hat{X}$. If $x, y \in X$ then there exist disjoint neighborhoods in $X$. So the only question is whether we can separate a point $x \in X$ from $\infty$. Since $X$ is locally compact, there exists an open neighborhood $W$ of $x$ such that the closure (in $X$) $\bar{W}$ is compact. Then $(X - \bar{W}) \cup \{\infty\}$ is a neighborhood of $\infty$ in $\hat{X}$ that is disjoint from $W$.

(4) Show $X$ is dense in $\hat{X}$ (unless $X$ was compact to start with). If $X$ is already compact, then the topology $\hat{\mathcal{T}}$ includes the singleton $\{\infty\}$, isolated from $X$. Assume now that $X$ is not compact. Let $U_C$ be a neighborhood of $\infty$. Since $X$ is not compact, in particular $X \neq C$, there must be some point(s) $x \in X - C$. That is, the neighborhood $U_C$ of $\infty$ contains point(s) of $X$.

□

**Final remarks.** We just showed that each locally compact Hausdorff space can be viewed as an open subset of a compact Hausdorff space. Conversely, each open subset of a compact Hausdorff space is itself a locally compact Hausdorff space (text Cor. 29.3). (more on this in class...)

(end of handout)