

Some highlights on
Countability and Separation Properties

(Relates to text Sec. 30–36)

Introduction. There are two basic themes to the next several sections:

- a. What properties of a topology allow us to conclude that the topology is given by a metric?
- b. What properties of a space allow us to conclude that the space actually is (homeomorphic to) a subspace of \mathbb{R}^n (or at least a subspace of R^ω)?

Countability Properties. Here are several properties of spaces, all saying that the topology, or some key feature of it, can be described in terms of countably many pieces of information. The names are historical; they are not very descriptive or otherwise useful, but you should know them since they are used in the literature.

- (1) **“First axiom of countability”** The space (X, \mathcal{T}) is called *first-countable* if the topology has a countable local basis at each point $x \in X$.
- (2) **“Second axiom of countability”** The space (X, \mathcal{T}) is called *second-countable* if the topology \mathcal{T} has a countable basis.
- (3) **“Separable”** The space (X, \mathcal{T}) is called *separable* if X contains a countable dense subset. Recall a subset $A \subseteq X$ is called *dense* in X if the closure \bar{A} is all of X , i.e. each open set contains at least one point of A .
- (4) **Lindelöf property** The space (X, \mathcal{T}) is called a *Lindelöf* space if each open cover of X has a countable sub-cover.

The familiar space \mathbb{R}^n , with the standard topology has all of the above properties (proof below). For more general spaces, we can ask many questions:

- Do any of these properties imply others?
- If a space X has one of the properties, do all subspaces of X have the property? (In that always happens, we would call the property *hereditary*.)
- If we have a family of spaces with one of these properties, does the cartesian product have the property?
- If $f : X \rightarrow Y$ is a continuous surjection, and X has one of the properties, must Y also have the property?
- If (X, \mathcal{T}) has one of these properties, and \mathcal{T}' is a coarser [resp. finer] topology, must (X, \mathcal{T}') have the property?

We will focus on just some highlights.

Theorem (text 30.3). *Countable basis \implies all the other countability properties.*

Proof. Suppose \mathcal{B} is a countable basis for the topology on X .

- a. Countable local basis: Let $x \in X$ and let U be any neighborhood of x . Since \mathcal{B} is a basis for the topology, U is a union of elements of \mathcal{B} . Thus there exists an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. So the set \mathcal{B} is a countable local basis for each point $x \in X$.
- b. Separable: For each nonempty set $B \in \mathcal{B}$, pick a point $x_B \in B$. Since \mathcal{B} is countable, the set $\{x_B \mid B \in \mathcal{B}\}$ is countable. Since each open set is a union of elements of \mathcal{B} , each nonempty open set U contains at least one of the sets B and so $x_B \in U$. Thus $\{x_B \mid B \in \mathcal{B}\}$ is dense in X .
- c. Lindelöf : Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of X . We want to prove there exists a countable subcover, by somehow using the existence of a countable basis \mathcal{B} for the topology. For convenience (to make the exact argument a little simpler), assume that one of the sets U_α is actually the empty set. (Or adjoin one additional set $U_0 = \emptyset$ to the covering.) The idea of the proof is to use the elements of \mathcal{B} to “point to” certain special U_α ’s. Specifically, for each set $B \in \mathcal{B}$, we will select one set U_B from among the U_α ’s as follows: First ask if there exists at least one of the open sets U_α containing that set B . If not, let $U_B = U_0 = \emptyset$. If the set B is contained in some U_α , then pick one such U_α and call it U_B . We might pick the same U_α corresponding to several B ’s (because a given U_α usually contains many basis sets), but we have at most one U_α chosen for each B ; so the set $\{U_B : B \in \mathcal{B}\}$ is countable.

We now show that $\{U_B : B \in \mathcal{B}\}$ covers X . Let $x \in X$. We shall prove that at least one of the sets U_B contains x . Since the U_α ’s cover X , there is some U_α containing x . Since \mathcal{B} is a basis, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_\alpha$. Since that basis set B is contained in some U_α , B is one of the basis sets for which we chose a set $U_B \supseteq B$. So $x \in B \subseteq U_B$, in particular $x \in U_B$.

□

The previous theorem says that having a countable basis for the topology is the strongest of the countability properties. The next example shows that it is strictly stronger, that is the other properties do not imply it.

Example (ex 3, page 192). *The space \mathbb{R}_ℓ is first-countable, separable, and Lindelöf, but not second-countable.*

Proof. The details are given in the text; you should be able to prove \mathbb{R}_ℓ is first-countable and separable, and that it is not second-countable. You are not required to know the proof that \mathbb{R}_ℓ is Lindelöf .

□

The previous example shows some of the independence of the properties. However, in metric spaces, the first-countable, separable, and second-countable properties are equivalent.

Theorem (Exercise 5 page 194). *Suppose X is a metric space. Then*

- i. X has countable local bases at each point.
- ii. X separable $\implies X$ has a countable basis.
- iii. X Lindelöf $\implies X$ has a countable basis.

Proof.

- i. The idea of “countable local basis” is precisely a generalization of the balls of radius $1/n$ in metric spaces: The set $\{B(x, \frac{1}{n})\}$ is a local basis at x .
- ii. Let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense set in X . For each x_n , let \mathcal{B}_n be the set of all open balls centered at x_n with rational radius. Then the set \mathcal{B}_n is countable for each n , so the set $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \mathbb{N}\}$ is countable. We claim this set \mathcal{B} is a basis. The proof is an exercise in using the triangle inequality that the metric satisfies. (You can work out the details: here is the idea...) Take any open set $U \subseteq X$. We want to show that U is a union of our alleged basis elements. Let y be any point of U ; we shall show that there is one of our alleged basis sets B such that $y \in B \subseteq U$. The point y is contained in some ϵ ball inside U . Now look at an $\epsilon/100$ ball around y . This ball must contain some point x_n from our countable dense subset. So the distance from x_n to y is less than $\epsilon/100$. Then by picking a rational radius slightly larger than $\epsilon/100$, we can find a rational-radius ball centered at x_n , containing y , and contained in U .
- iii. For each n , consider the open covering of X consisting of all balls of radius $1/n$. The Lindelöf property says there exists a countable subcover \mathcal{B}_n . Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \mathbb{N}\}$. This is a countable union of countable sets, hence countable. Check that it is indeed a basis.

□

Remark. The the preceding proofs, it might seem that all we need is “separable + countable local basis” or “Lindelöf + countable local basis” to conclude that X has a countable basis. But remember the previous example of a space that IS separable, DOES have the Lindelöf property, DOES have a countable local basis at each point, but does not have a countable basis for the topology. The triangle inequality property of metrics provides an extra amount of “niceness” for the topology, so we can connect from one property to the other.

Now let us consider subspaces and cartesian products.

Theorem (30.2). *The properties countable local basis and countable basis are preserved for subspaces and for countable cartesian products.*

Proof. Proofs are given in the text.

□

Theorem. *The Lindelöf property is inherited by closed subspaces (analogous to compactness). (But it is not necessarily inherited by arbitrary subspaces; see Example 4 on page 193).*

Proof. This is Problem 9 on page 194. Consider it one of your possible problems for the Final Exam. □

Example. *The property of being separable need not be inherited by subspaces; having the subspace be closed seems irrelevant to this question.*

The space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable but the diagonal line $\{(x, -x) : x \in \mathbb{R}\}$ is a closed subspace that is homeomorphic to \mathbb{R} with the discrete topology. This uncountable closed discrete subspace makes \mathbb{R}_ℓ^2 a useful counterexample for several questions. It shows that being separable is not hereditary, the Lindelöf property is not always preserved by finite products, and [next section] the property of being normal is not always preserved by finite products.

Homework in Section 30. Page 194

Problems 2, 3, 4, 11, 13

Additional problems it would be good to study (consider these among the sample problems for the final exam): Problems 5, 6 (just \mathbb{R}_ℓ), 7, 9, 10, 14.

(end of handout)